Embeddings of decomposition spaces

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Function Spaces, Differential Operators and Nonlinear Analysis
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Outline

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$$\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q) := \left\{ f \in \;\; ??? \;\; \middle| \; \left(\left\| \mathcal{F}^{-1}(\varphi_i \cdot \widehat{f}) \right\|_{L^p} \right)_{i \in I} \in \ell_w^q(I) \right\}.$$

Note: $\ell_w^q(I) = \left\{ (c_i)_{i \in I} \in \mathbb{C}^I \,\middle|\, (w_i \cdot c_i)_{i \in I} \in \ell^q(I) \right\}$.

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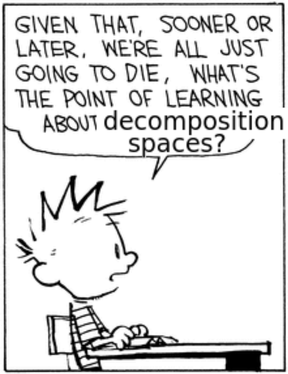
But... WHY?


GIVEN THAT, SOONER OR LATER, WE'RE ALL JUST GOING TO DIE, WHAT'S THE POINT OF LEARNING ABOUT decomposition spaces?

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The class of decomposition spaces looks like this:

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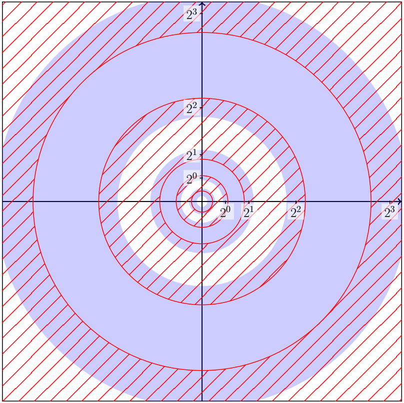
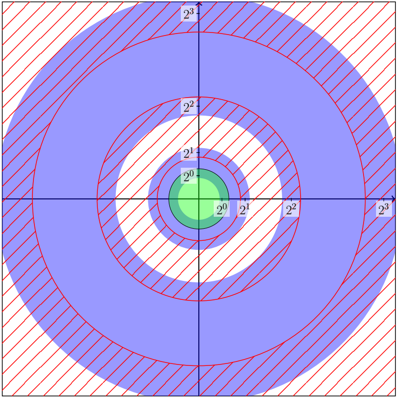
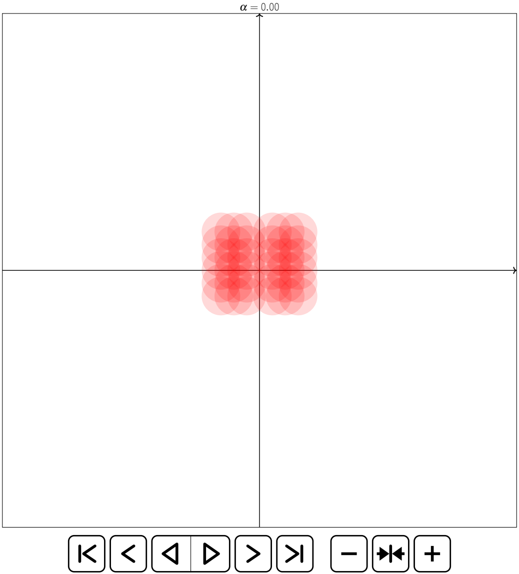
How do the associated coverings look like?

Besov spaces

homogeneous	inhomogeneous
	

α-modulation spaces



Wavelet-type coorbit spaces

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Theorem (Führ, FV; 2014)

Let $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$ *such that the quasi-regular representation*

$$[\pi(x,h)f](y) = |\det h|^{1/2} \cdot f(h(y-x))$$

of $\mathbb{R}^d \rtimes H$ *on* $L^2\left(\mathbb{R}^d\right)$ *is irreducible and square-integrable.*

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Then, for a suitable weight $w = w_{m,q}$,

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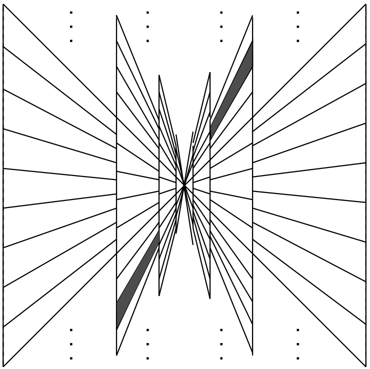

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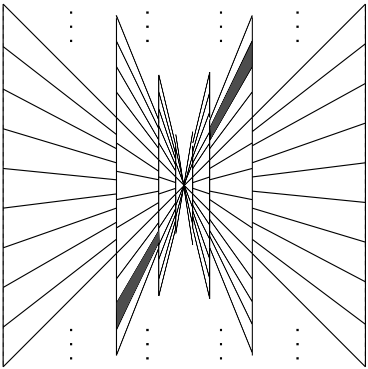
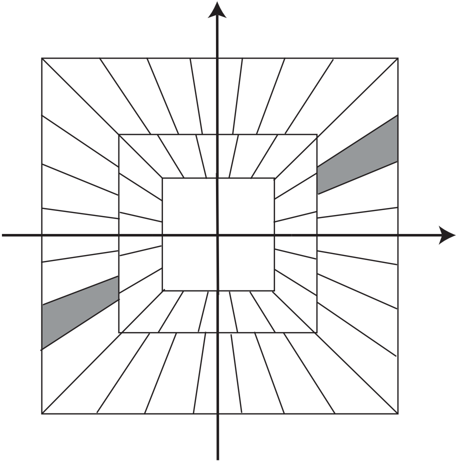


We can study coorbit spaces (e.g. embeddings between coorbit spaces defined on different groups) using decomposition space theory.

Shearlet smoothness spaces



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And what do we mean by that?

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The covering $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I}$ is **relatively moderate with respect to** $\mathcal{P}$ if the weight $(|\det T_i|)_{i \in I}$ is relatively $\mathcal{P}$-moderate.

Roughly: If the "small" sets Q_i, Q_ℓ intersect the same "large" set P_j, then $\lambda(Q_i) \asymp \lambda(Q_\ell)$.

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Theorem (FV; 2015)

If
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$$\iota : \quad \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), \quad g \mapsto \textstyle\sum_{i \in I} \varphi_i g$$

is bounded and $\iota g \in \mathcal{D}'(\mathcal{O}')$ *extends* $g \in \mathcal{D}_{\mathcal{F}}\left(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}\right) \leq \mathcal{D}'(\mathcal{O})$.

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it is sufficient for the embedding if

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Conversely, if $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and if

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Improvements for relatively moderate coverings

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*Under the above assumptions, $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$ holds **if and only if** if we have $p_1 \leq p_2$ and $(\bigstar_{p_2^{\triangledown}}) < \infty$.*

Sample application

When do we have

$$M_{s_1}^{p_1,q_1}(\mathbb{R}^d) \overset{?}{\hookrightarrow} B_{s_2}^{p_2,q_2}(\mathbb{R}^d) \qquad (\bigstar)$$

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$$(\bigstar) \Leftrightarrow (p_1 \leq p_2) \quad \text{and} \quad \begin{cases} s_2 \leq s_1 - d\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+, & \text{if } q_1 \leq q_2, \\ s_2 < s_1 - d\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+, & \text{if } q_1 > q_2. \end{cases}$$

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We provided a **general framework** for embeddings

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If you need an embedding for decomposition spaces, first try the framework!

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Literature:

- F. Voigtlaender, Embeddings of decomposition spaces, arXiv:1605.09705
- F. Voigtlaender, Embeddings of Decomposition Spaces into Sobolev and BV spaces, arXiv:1601.02201
- F. Voigtlaender, Embedding theorems for decomposition spaces with applications to wavelet coorbit spaces, PhD thesis

Thank you!

Thank you!

Questions, comments, counterexamples?

☺

Outline

Sufficient criterion if $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$

Let

$$(\blacksquare_r) := \left\| \left(w_i^{-1} \cdot \left\| (v_j / u_{i,j})_{j \in J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}},$$

where

$$u_{i,j} := \begin{cases} |\det S_j|^{p_2^{-1} - 1} \cdot |\det T_i|^{1 - p_1^{-1}}, & \text{if } p_1 < 1, \\ |\det S_j|^{p_2^{-1} - p_1^{-1}}, & \text{if } p_1 \geq 1. \end{cases}$$

Theorem (FV)

If • $\mathcal{P}$ *is almost subordinate to* $\mathcal{Q}$,

- $p_1 \leq p_2$,
- $(\blacksquare_{p_1^{\triangle}}) < \infty$, *where* $p_1^{\triangle} := \max\{p_1, p_1'\}$,

then $\mathcal{O}' \subset \mathcal{O}$ *and the map*

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \to \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), f \mapsto f|_{C_c^\infty(\mathcal{O}')}$$

is bounded.

Let $\mathcal{P}$ be almost subordinate to $\mathcal{Q}$. Let $1/p_1^{\pm\triangle} = \min\left\{p_1^{-1}, 1 - p_1^{-1}\right\}$.

If
$$\left(C_c^\infty\left(\mathcal{O}'\right), \left\|\cdot\right\|_{\mathcal{D}_{\mathcal{F}}\left(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}\right)}\right) \hookrightarrow \mathcal{D}_{\mathcal{F}}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), g \mapsto g$$
is bounded, then $p_1 \leq p_2$ and $(\blacksquare_{p_1}^*) < \infty$, where
$$(\blacksquare_r^*) := \left\|\left(w_i^{-1} \cdot \left\|\left(|\det S_j|^{p_1^{-1} - p_2^{-1}} \cdot v_j\right)_{j \in J_i}\right\|_{\ell^{q_2 \cdot (r/q_2)'}}\right)_{i \in I}\right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}.$$

If $\mathcal{P}$ and v are relatively $\mathcal{Q}$-moderate, then
$$\mathcal{D}_{\mathcal{F}}\left(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}\right) \hookrightarrow \mathcal{D}_{\mathcal{F}}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$$
is **equivalent** to $p_1 \leq p_2$ and
$$\left\|\left(w_i^{-1} \cdot v_{j_i} \cdot |\det T_i|^s \cdot |\det S_{j_i}|^{p_1^{-1} - p_2^{-1} - s}\right)_{i \in I}\right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} < \infty,$$
where $s := \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+$ and where each $j_i \in J_i$ can be selected arbitrarily.

Embeddings for α-modulation spaces

Let $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$ and define

$$\gamma^{(0)} := \alpha \left(p_2^{-1} - p_1^{-1}\right) + (\alpha - \beta) \left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+ ,$$

$$\gamma^{(1)} := \alpha \left(p_2^{-1} - p_1^{-1}\right) + (\alpha - \beta) \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ .$$

We have $M^{p_1,q_1}_{\gamma_1,\alpha}\left(\mathbb{R}^d\right) \hookrightarrow M^{p_2,q_2}_{\gamma_2,\beta}\left(\mathbb{R}^d\right)$ if and only if $p_1 \leq p_2$ and

$$\begin{cases} \gamma_2 \leq \gamma_1 + d\gamma^{(0)}, & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d\left(\gamma^{(0)} + (1-\beta)\left(q_1^{-1} - q_2^{-1}\right)\right), & \text{if } q_1 > q_2. \end{cases}$$

We have $M^{p_1,q_1}_{\gamma_1,\beta}\left(\mathbb{R}^d\right) \hookrightarrow M^{p_2,q_2}_{\gamma_2,\alpha}\left(\mathbb{R}^d\right)$ if and only if $p_1 \leq p_2$ and

$$\begin{cases} \gamma_2 \leq \gamma_1 + d\gamma^{(1)}, & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d\left(\gamma^{(1)} + (1-\beta)\left(q_1^{-1} - q_2^{-1}\right)\right), & \text{if } q_1 > q_2. \end{cases}$$

Embeddings for Shearlet smoothness spaces

We have

$$S_\beta^{p_1,q_1}(\mathbb{R}^2) \hookrightarrow B_\alpha^{p_2,q_2}(\mathbb{R}^2)$$

if and only if $p_1 \le p_2$ and

$$\begin{cases} \alpha \le \beta - \frac{3}{2}\left(p_1^{-1} - p_2^{-1}\right) - \frac{1}{2}\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+, & \text{if } q_1 \le q_2, \\ \alpha < \beta - \frac{3}{2}\left(p_1^{-1} - p_2^{-1}\right) - \frac{1}{2}\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+, & \text{if } q_1 > q_2. \end{cases}$$

We have

$$B_\alpha^{p_1,q_1}(\mathbb{R}^2) \hookrightarrow S_\beta^{p_2,q_2}(\mathbb{R}^2)$$

if and only if $p_1 \le p_2$ and

$$\begin{cases} \beta \le \alpha - \frac{1}{2}\left(q_2^{-1} - \frac{1}{p_1^{\pm\triangle}}\right)_+ - \frac{3}{2}\left(p_1^{-1} - p_2^{-1}\right), & \text{if } q_1 \le q_2, \\ \beta < \alpha - \frac{1}{2}\left(q_2^{-1} - \frac{1}{p_1^{\pm\triangle}}\right)_+ - \frac{3}{2}\left(p_1^{-1} - p_2^{-1}\right), & \text{if } q_1 > q_2. \end{cases}$$

We have $\dot{B}_{\alpha}^{p_1,q_1}\left(\mathbb{R}^d\right) \hookrightarrow B_{\beta}^{p_2,q_2}\left(\mathbb{R}^d\right)$ if $p_1 \leq p_2$,

$$\begin{cases} \alpha \leq d\left(p_1^{-1} - p_2^{-1}\right) \leq 0, & \text{if } q_1 \leq p_2^{\triangledown}, \\ \alpha < d\left(p_1^{-1} - p_2^{-1}\right) \leq 0, & \text{if } q_1 > p_2^{\triangledown} \end{cases}$$

and

$$\begin{cases} \beta \leq \alpha + d\left(p_2^{-1} - p_1^{-1}\right), & \text{if } q_1 \leq q_2, \\ \beta < \alpha + d\left(p_2^{-1} - p_1^{-1}\right), & \text{if } q_1 > q_2. \end{cases} \tag{1}$$

Conversely, if $\dot{B}_{\alpha}^{p_1,q_1}\left(\mathbb{R}^d\right) \hookrightarrow B_{\beta}^{p_2,q_2}\left(\mathbb{R}^d\right)$, then $p_1 \leq p_2$ and

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and equation (1) holds.

Embeddings between Besov spaces (2)

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as well as

$$\begin{cases} \beta \geq d\left(p_2^{-1} - p_1^{-1}\right), & \text{if } q_2 \geq p_1^{\triangle} \text{ and } p_1 \in [1,\infty] \\ \beta \geq d\left(p_2^{-1} - 1\right), & \text{if } q_2 = \infty \text{ and } p_1 \in (0,1), \\ \beta > d\left(p_2^{-1} - p_1^{-1}\right), & \text{if } q_2 < p_1^{\triangle} \text{ and } p_1 \in [1,\infty], \\ \beta > d\left(p_2^{-1} - 1\right), & \text{if } q_2 < \infty \text{ and } p_1 \in (0,1). \end{cases}$$

Conversely, if $B_{\alpha}^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}_{\beta}^{p_2,q_2}(\mathbb{R}^d)$, then $p_1 \leq p_2$, eq. (2) holds and

$$\begin{cases} \beta \geq d\left(p_2^{-1} - p_1^{-1}\right), & \text{if } q_2 \geq p_1, \\ \beta > d\left(p_2^{-1} - p_1^{-1}\right), & \text{if } q_2 < p_1, \\ \beta \geq d\left(p_2^{-1} - 1\right), & \text{if } q_2 = \infty, \\ \beta > d\left(p_2^{-1} - 1\right), & \text{if } q_2 < \infty. \end{cases}$$

Embeddings of shearlet coorbit spaces into Besov spaces

Let $c \in (0,1]$ and $u^{(\alpha,\beta)} : H_c \to (0,\infty), h \mapsto \left\| h^{-1} \right\|^{\alpha} \cdot |\det h|^{\beta}$, as well as

$$\alpha^{(1)} := \frac{1+c}{c} \cdot \left(p_1^{-1} - p_2^{-1} - q_1^{-1} + \beta + 1/2\right),$$

$$\gamma^{(1)} := -(1+c)\left(p_1^{-1} - p_2^{-1} - q_1^{-1} + \beta + 1/2\right) + (c-1)\left(\frac{1}{p_2^{\triangledown}} - q_1^{-1}\right)_+ ,$$

If $p_1 \leq p_2$ and

$$\begin{cases} \max\left\{\alpha, \frac{1}{p_2^{\triangledown}} - q_1^{-1}\right\} < \alpha^{(1)}, & \text{if } q_1 > p_2^{\triangledown}, \\ \max\{\alpha, 0\} \leq \alpha^{(1)}, & \text{if } q_1 \leq p_2^{\triangledown}, \end{cases}$$

as well as

$$\begin{cases} \gamma \leq \alpha + \gamma^{(1)}, & \text{if } q_1 \leq q_2 \\ \gamma < \alpha + \gamma^{(1)}, & \text{if } q_1 > q_2, \end{cases}$$

then

$$\mathrm{Co}\left(L^{p_1,q_1}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \hookrightarrow B^{p_2,q_2}_{\gamma}\left(\mathbb{R}^2\right).$$

These conditions are necessary, if $p_2^{\triangledown}$ is replaced by p_2 everywhere.

Embeddings of Besov spaces into shearlet coorbit spaces

Let $c \in (0,1]$ and $u^{(\alpha,\beta)} : H_c \to (0,\infty), h \mapsto \left\| h^{-1} \right\|^{\alpha} \cdot |\det h|^{\beta}$, as well as

$$\alpha^{(2)} := \frac{1+c}{c} \cdot \left(p_2^{-1} - \min\left\{1, p_1^{-1}\right\} - q_2^{-1} + \beta + 1/2\right),$$

$$\gamma^{(2)} := (1+c)\left(\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{q_2} - \beta - \frac{1}{2}\right) + (1-c)\left(\frac{1}{q_2} - \min\left\{\frac{1}{p_1}, 1 - \frac{1}{p_1}\right\}\right)_+ .$$

If we have $p_1 \leq p_2$ and

$$\begin{cases} \min\left\{\alpha, \frac{1}{p_1^{\triangle}} - \frac{1}{q_2}\right\} > \alpha^{(2)}, & \text{if } p_1^{\triangle} > q_2, \\ \min\{\alpha, 0\} \geq \alpha^{(2)}, & \text{if } p_1^{\triangle} \leq q_2, \end{cases}$$

as well as

$$\begin{cases} \gamma \geq \alpha + \gamma^{(2)}, & \text{if } q_1 \leq q_2, \\ \gamma > \alpha + \gamma^{(2)}, & \text{if } q_1 > q_2, \end{cases}$$

then $B_{\gamma}^{p_1,q_1}\left(\mathbb{R}^2\right) \hookrightarrow \mathrm{Co}\left(L_{u^{(\alpha,\beta)}}^{p_2,q_2}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$.

These conditions are necessary, if $p_1^{\pm\triangle}$ and $p_1^{\triangle}$ are replaced by p_1 everywhere.