Well-posed space-time variational formulations of evolutionary PDEs with applications

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Topics

- Adaptive wavelet methods for solving linear and nonlinear operator equations
- Application to space-time variational formulation of parabolic PDEs
- Application to space-time variational formulation of instat. (N)SE

Intoduction & motivation

Consider $\begin{cases} -\triangle u = f & \text{on } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$ or, in variational form: $u \in H_0^1(\Omega)$ s.t.

$$\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, dx = \int_\Omega f v \, dx \quad (v \in H_0^1(\Omega)).$$

For closed subsp. $S \subset H_0^1(\Omega)$, solve problem on S (**Galerkin**), giving best approximation from S w.r.t. $|\cdot|_{H^1(\Omega)}$.

Take S cont. piecewise pols of order $p \geq 2$, zero on bdr, w.r.t. a triangulation of Ω (**finite element space**). Then for a **quasi-uniform** triangulation, with $N := \dim S$,

$$|u - u_S|_{H^1(\Omega)} \lesssim N^{-\frac{p-1}{n}} |u|_{H^p(\Omega)}.$$

On a polygon with maximal interior angle α ($\in [\frac{\pi}{3}, 2\pi]$), $u \in H^s(\Omega)$ for $s < 1 + \frac{\pi}{\alpha}$ only. Rate $\frac{p-1}{n}$ reduces to $\frac{s-1}{n}$.
Optimal rate $\frac{p-1}{n}$ can be retrieved by **proper local refinements** when $u \in B^p_{\tau,q}(\Omega)$ for $\tau > (\frac{1}{2} + \frac{p}{2})^{-1}$ ([BDDP02]); and, for sufficiently smooth f, latter holds true ([DD97]).

Wavelets

Adaptive methods aim at constructing iteratively a seq. of **quasi-optimal** meshes based on **a posteriori** information. Instead of adaptive fem, talk is about adaptive **wavelet** methods. Now S is spanned by (adaptively chosen) (nested) subsets of wavelet basis for $H_0^1(\Omega)$. **Not** restricted to **elliptic** problems.

• Orthogonal wavelet basis $\Psi = \{\psi_\lambda \colon \lambda \in \nabla\}$ for $L_2(0,1)$ (properly scaled, Riesz basis for $H^s(0,1)$ for $s \in (-\frac{1}{2}, \frac{1}{2})$):

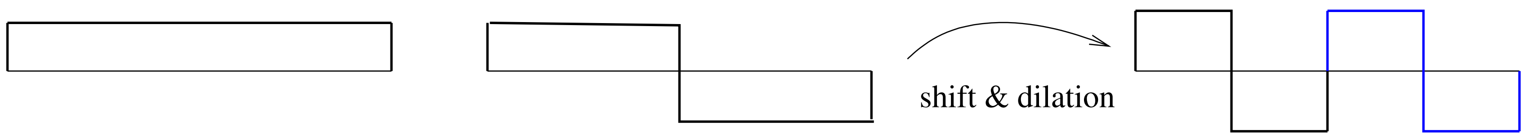
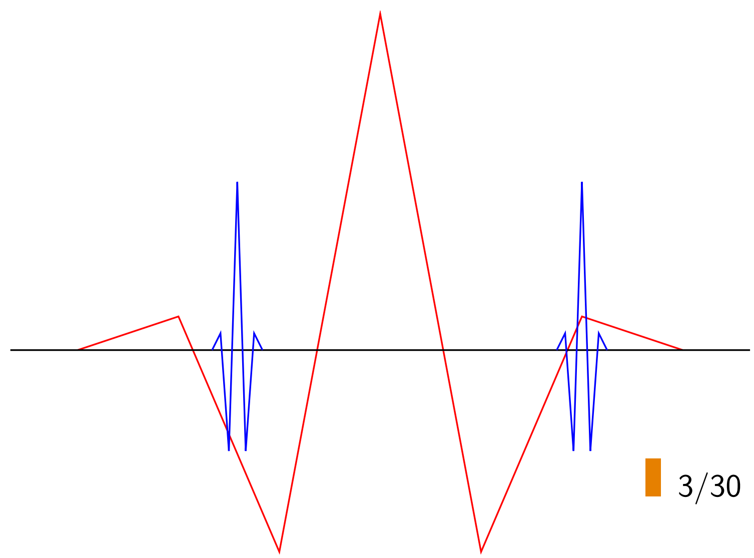


Figure 1: Haar wavelets ψ_λ with **levels** $|\lambda| = 0, 1, ..,\ \operatorname{diam}\operatorname{supp}\psi_\lambda \eqsim 2^{-|\lambda|}$

• Riesz basis of continuous piecewise polynomial wavelets for $H^s(\Omega)$ ($H_0^s(\Omega)$) for $s \in (-s_0, \frac{3}{2})$ (also on polytopes in $n > 1$ dimensions):



Setting: Well-posed (lin.) op. eqs.

Let $\mathcal{X}$, $\mathcal{Y}$ be sep. Hilbert spaces (over $I\!R$). Let $B \in \mathcal{L}\mathrm{is}(\mathcal{X}, \mathcal{Y}')$. Given $f \in \mathcal{Y}'$, we seek $u \in \mathcal{X}$ s.t.

$$Bu = f.$$

Ex.:

- $(Bw)(v) := \int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, v$, $\mathcal{X} = \mathcal{Y} := H_0^1(\Omega)$ (Poisson problem),
- $(B(\vec{w}, p))(\vec{v}, q) := \int_\Omega \mathbf{grad}\, \vec{w} : \mathbf{grad}\, \vec{v} - \int_\Omega p \operatorname{div} \vec{v} - \int_\Omega q \operatorname{div} \vec{w}$, $\mathcal{X} = \mathcal{Y} := H_0^1(\Omega)^n \times L_2(\Omega)/I\!R$ for a domain $\Omega \subset I\!R^n$ (stat. Stokes problem),
- $(Bw)(v) := \frac{1}{4\pi} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(w(y)-w(x))(v(y)-v(x))}{|x-y|^3} dx dy$, $\Omega \subset I\!R^3$, $\mathcal{X} = \mathcal{Y} := H^{\frac{1}{2}}(\partial\Omega)/\mathbb{R}$ (hypersingular boundary integral equation).
- ODE's, parabolic problems, instat. Stokes: $\mathcal{X} \neq \mathcal{Y}$.

Reformulation as well-posed bi-infinite MV eq

Let $\Psi^{\mathcal{X}} = \{\psi_\lambda^{\mathcal{X}} : \lambda \in \nabla_{\mathcal{X}}\}$, $\Psi^{\mathcal{Y}} = \{\psi_\lambda^{\mathcal{Y}} : \lambda \in \nabla_{\mathcal{Y}}\}$ Riesz bases for $\mathcal{X}$, $\mathcal{Y}$ (we have wavelet bases in mind). That is, the **synthesis operator**,

$$\mathcal{F}_{\mathcal{X}} : \mathbf{c} \mapsto \mathbf{c}^\top \Psi^{\mathcal{X}} := \sum_{\lambda \in \nabla_{\mathcal{X}}} c_\lambda \psi_\lambda^{\mathcal{X}} \in \mathcal{L}\mathrm{is}(\ell_2(\nabla_{\mathcal{X}}), \mathcal{X}),$$

and so its adjoint, the **analysis operator**,

$$\mathcal{F}'_{\mathcal{X}} : g \mapsto g(\Psi^{\mathcal{X}}) := [g(\psi_\lambda^{\mathcal{X}})]_{\lambda \in \nabla_{\mathcal{X}}} \in \mathcal{L}\mathrm{is}(\mathcal{X}', \ell_2(\nabla_{\mathcal{X}})).$$

(analogously for $\mathcal{F}_{\mathcal{Y}}$).

$$Bu = f \iff \underbrace{\mathcal{F}'_{\mathcal{Y}} B \mathcal{F}_{\mathcal{X}}}_{\mathbf{B}} \underbrace{\mathcal{F}_{\mathcal{X}}^{-1} u}_{\mathbf{u}} = \underbrace{\mathcal{F}'_{\mathcal{Y}} f}_{\mathbf{f}},$$

where

$$\mathbf{B} = (B\Psi^{\mathcal{X}})(\Psi^{\mathcal{Y}}) \in \mathcal{L}\mathrm{is}(\ell_2(\nabla_{\mathcal{X}}), \ell_2(\nabla_{\mathcal{Y}}))$$

(infinite "stiffness" matrix),

$$\mathbf{f} = f(\Psi^{\mathcal{Y}}) \in \ell_2(\nabla) \quad \text{(infinite "load" vector).}$$

Adaptive Wavelet-Galerkin scheme ($\mathbf{Bu} = \mathbf{f}$)

([CDD01]) Let $\mathscr{X} = \mathscr{Y}$, $\Psi^{\mathscr{X}} = \Psi^{\mathscr{Y}}$, and $B = B' > 0$, so that $\mathbf{B} = \mathbf{B}^\top > 0$. Otherwise apply the following to $\mathbf{B}^\top\mathbf{B}\mathbf{u} = \mathbf{B}^\top\mathbf{f}$.

Goal: To generate sequence of approx. to $\mathbf{u}$ that, whenever for some $s > 0$, $\|\mathbf{u}\|_{\mathcal{A}^s} := \sup_N N^s \|\mathbf{u} - \mathbf{u}_N\| < \infty$, converges with **this** rate s, at linear cost. (Here $\mathbf{u}_N$ is a best approx. to $\mathbf{u}$ with $\#\,\mathrm{supp}\,\mathbf{u}_N \leq N$.)

Notations: $\Lambda \subseteq \nabla$, $\mathbf{I}_\Lambda : \ell_2(\Lambda) \to \ell_2(\nabla)$, $\mathbf{R}_\Lambda = \mathbf{I}_\Lambda^\top : \ell_2(\nabla) \to \ell_2(\Lambda)$,

$$\mathbf{B}_\Lambda := \mathbf{R}_\Lambda \mathbf{B} \mathbf{I}_\Lambda, \quad \mathbf{u}_\Lambda := \mathbf{B}_\Lambda^{-1} \mathbf{R}_\Lambda \mathbf{f}, \quad ||| \cdot ||| := \langle \mathbf{B} \cdot, \cdot \rangle^{\frac{1}{2}}$$

(we will identify $\mathbf{u}_\Lambda$ with $\mathbf{I}_\Lambda \mathbf{u}_\Lambda$).

awgm:
- Solve $\mathbf{B}_{\Lambda_i} \mathbf{u}_{\Lambda_i} = \mathbf{R}_{\Lambda_i} f$;
- $\Lambda_{i+1} \supset \Lambda_i$ by bulk chasing on $\mathbf{f} - \mathbf{B}\mathbf{u}_{\Lambda_i}$;
- repeat with $i := i + 1$

Prop 1 ([CDD01]). *Let* $\theta \in (0,1]$, $\Lambda \subset \Xi \subset \nabla$, *s.t.*

$$\|\mathbf{R}_\Xi(\mathbf{f} - \mathbf{B}\mathbf{u}_\Lambda)\| \geq \theta \|\mathbf{f} - \mathbf{B}\mathbf{u}_\Lambda\|.$$

Then $|||\mathbf{u} - \mathbf{u}_\Xi||| \leq \left[1 - \kappa(\mathbf{B})^{-1}\theta^2\right]^{\frac{1}{2}} |||\mathbf{u} - \mathbf{u}_\Lambda|||.$

Prop 2 ([GHS07]). *If* $\theta < \kappa(\mathbf{B})^{-\frac{1}{2}}$ *and* Ξ *is the* **smallest** *set satisfying bulk chasing criterium, then* $\#(\Xi \backslash \Lambda) \leq N$ *for smallest* N *s.t.*

$$|||\mathbf{u} - \mathbf{u}_N||| \leq [1 - \theta^2 \kappa(\mathbf{B})]^{\frac{1}{2}} |||\mathbf{u} - \mathbf{u}_\Lambda|||.$$

Corol 1. awgm *realizes optimal rate* s,

but, in this form, it is not implementable.

Thm 1. awgm *with* **approx.** *eval. of residual* $\mathbf{f} - \mathbf{B}\mathbf{u}_\Lambda$ *and* **approx.** *solution of* $\mathbf{B}_\Lambda \mathbf{u}_\Lambda = \mathbf{R}_\Lambda \mathbf{f}$ *within suff. small, but* **fixed rel. tolerance** δ, *also converges with optimal rate* s,
and, if such approx. residual eval. takes $\mathcal{O}(\|\mathbf{u} - \mathbf{u}_\Lambda\|^{-1/s} + \#\Lambda)$ *operations* (**cost condition**), *then scheme has* **optimal comput. compl.**

Nonlinear operator equations

([XZ03, Ste14]) Theorem 1 generalizes to equations

$$F(u) = 0$$

for $F : \mathcal{X} \supset \mathrm{dom}(F) \to \mathcal{Y}'$, written as

$$\mathbf{F}(\mathbf{u}) = \mathbf{0},$$

where $\mathbf{F} := \mathcal{F}'_{\mathcal{Y}} F \mathcal{F}_{\mathcal{X}}$, assuming that $\mathcal{X} = \mathcal{Y}$, $DF(u) \in \mathcal{L}\mathrm{is}(\mathcal{X}, \mathcal{X}')$ and $DF(u) = DF(u)' > 0$,

or

written as

$$D\mathbf{F}(\mathbf{u})^\top \mathbf{F}(\mathbf{u}) = \mathbf{0},$$

only assuming that $DF(u) \in \mathcal{L}\mathrm{is}(\mathcal{X}, \mathcal{Y}')$.

Cost condition

Approx. res. eval. scheme from [CDD01] exploits **near-sparsity** of $\mathbf{f}$ and $\mathbf{B}$, consequences of the **vanishing moments**, **and** that of $\mathbf{u}_{\Lambda_i}$ ($\|\mathbf{u}_{\Lambda_i}\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}$).

Satisfies cost condition, but is quantitatively costly.

A much more efficient scheme is possible when the application of operator to a wavelet 'lands' in L_2. It generalizes to nonlinear operators.
Since construction of wavelets with required smoothness is cumbersome in more than one dimensions, we advocate to write a well-posed 2nd order operator equation as a well-posed 1st order least squares system. Always possible.

Application: Parabolic problems

$\Omega \subset \mathbb{R}^n$, $\mathrm{I} = (0, T)$.

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle u = f & \text{on } \mathrm{I} \times \Omega, \\ u = 0 & \text{on } \mathrm{I} \times \partial\Omega, \\ u(0, \cdot) = 0 & \text{on } \Omega. \end{cases}$$

- $-\triangle$ can be read as semi-linear elliptic operator.
- other (inhom) initial or boundary conditions are allowed.

Standard appr.: Approx. $\frac{\partial u}{\partial t}(t, \cdot)$ by, say $\frac{u(t,\cdot)-u(t-h,\cdot)}{h}$, and solve seq. of elliptic problems for $0 < t_1 < t_2 < \cdots < t_M = T$

$$\begin{cases} -\triangle u(t_i, \cdot) - (t_i - t_{i-1})^{-1} u(t_i, \cdot) = (t_i - t_{i-1})^{-1} u(t_{i-1}, \cdot) + f(t_i, \cdot) & \text{on } \Omega \\ u(t_i, \cdot) = 0 & \text{on } \partial\Omega \end{cases}$$

- How to distribute optimally 'grid points' over space and time?
- Even if you can, approximation not effective for singularities that are local in space and time.
- Inherently sequential.
- When the whole time evolution is needed, as with problems of optimal control or in visualizations, huge amount of storage.

Space-time variational formulation

$(Bu)(v) := \int_I \int_\Omega \frac{\partial u}{\partial t} v + \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, dx dt = \int_I \int_\Omega f v \, dx dt := f(v).$

$B \in \mathcal{L}\mathrm{is}\Big(\underbrace{L_2(\mathrm{I}; H^1_0(\Omega)) \cap H^1_{0,\{0\}}(\mathrm{I}; H^{-1}(\Omega))}_{\mathscr{U} :=}, \underbrace{L_2(\mathrm{I}; H^1_0(\Omega))}_{\mathscr{V} :=}{}'\Big)$ (e.g. [LM70]).

After selecting Riesz $\Psi^{\mathscr{U}}$, $\Psi^{\mathscr{V}}$ for $\mathscr{U}$, $\mathscr{V}$, apply **awgm** to $\mathbf{B}^\top(\mathbf{B}\mathbf{u} - \mathbf{f}) = 0$, where $\mathbf{B} := (B\Psi^{\mathscr{U}})(\Psi^{\mathscr{V}})$, $\mathbf{f} := f(\Psi^{\mathscr{V}})$.

(even better first to write it as a well-posed first order system).

- Since $\mathscr{U}$ and $\mathscr{V}$ are (intersections of) tensor products of Hilbert spaces of temporal and spatial functions, they can be equipped with **tensor products** of temporal and spatial wavelet collections.

- Conseq., for suff. smooth sol, rate of best N-term approximation **equals** that for the corresponding stationary problem (cf. **sparse grids** or **hyperbolic cross** approx.).

  (Charac. of approx. classes as tensor product of Besov spaces (cf. [Nit06, HS12]), and corresponding reg. theory known for the elliptic case seem open.)

First test on an ODE

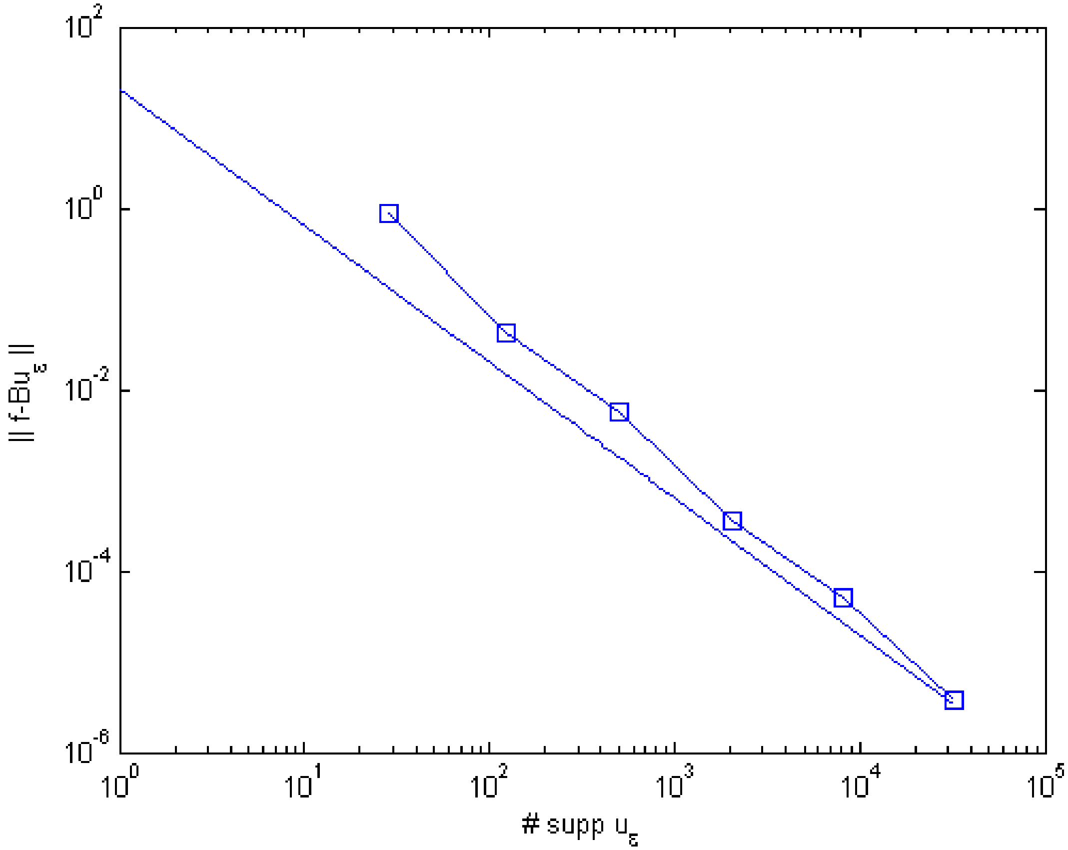
$$\begin{cases} \frac{du(t)}{dt} + \nu u(t) = g(t) & (t \in \mathrm{I}), \\ u(0) = u_0, & \end{cases}$$

$$(B_\nu)(v) := \int_\mathrm{I} -u(t)\frac{dv(t)}{dt} + \nu u(t)v(t)dt, \quad f(v) := \int_\mathrm{I} g(t)v(t)dt + u_0 v(0).$$

Prop 3. *With* $\mathcal{X} := L_2(\mathrm{I})$ *and* $\mathcal{Y}_\nu := H^1_{0,\{T\}}(\mathrm{I})$, *equipped with* $\|\cdot\|_{\mathcal{Y}_\nu} := \sqrt{\nu^2\|\cdot\|^2_{L_2(\mathrm{I})} + |\cdot|^2_{H^1(\mathrm{I})}}$, *the operator* $B_\nu \in \mathcal{L}\mathrm{is}(\mathcal{X}, \mathcal{Y}'_\nu)$ *and* $\kappa(B_\nu) \leq 2$.

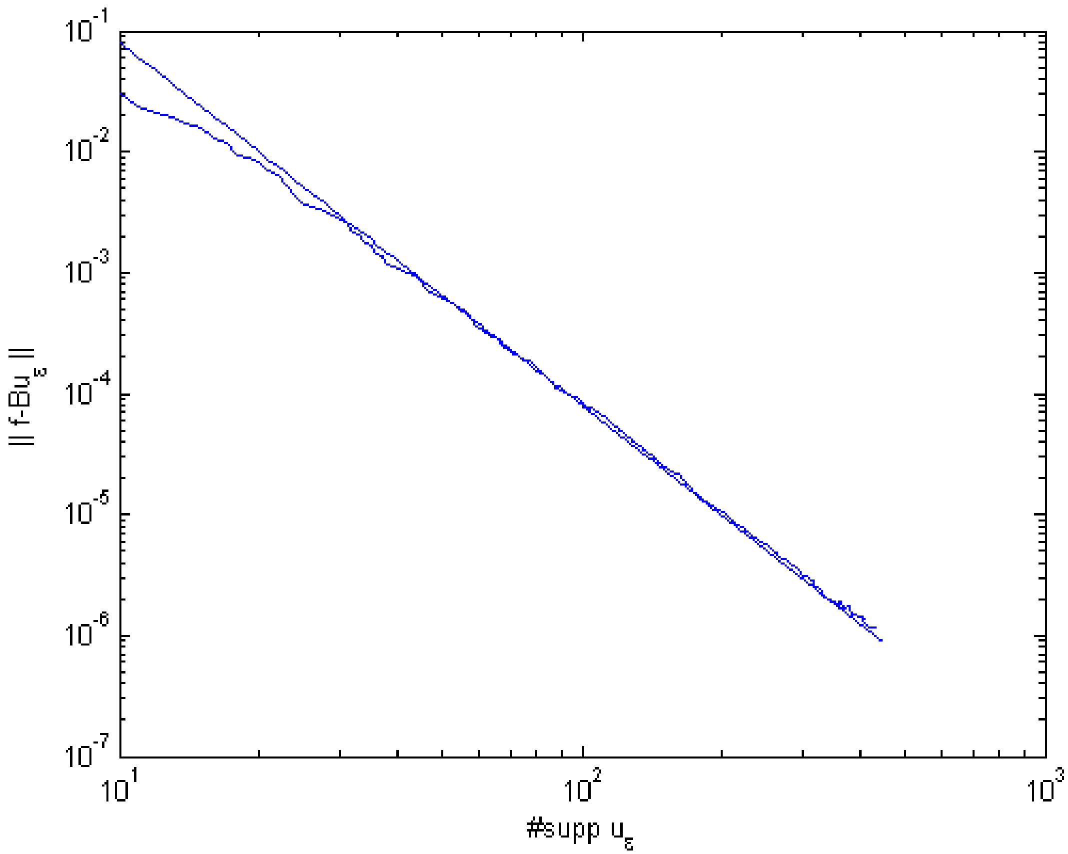
Num. results for $\nu = 1$, $g = 1$ on $(0, \frac{1}{3})$, $g = 2$ on $(\frac{1}{3}, 1)$, $u_0 = 0$ or $u_0 = 1$:

Uniform, non-adaptive refinements, i.e. collect all wavelets (of order 3) up to some level.



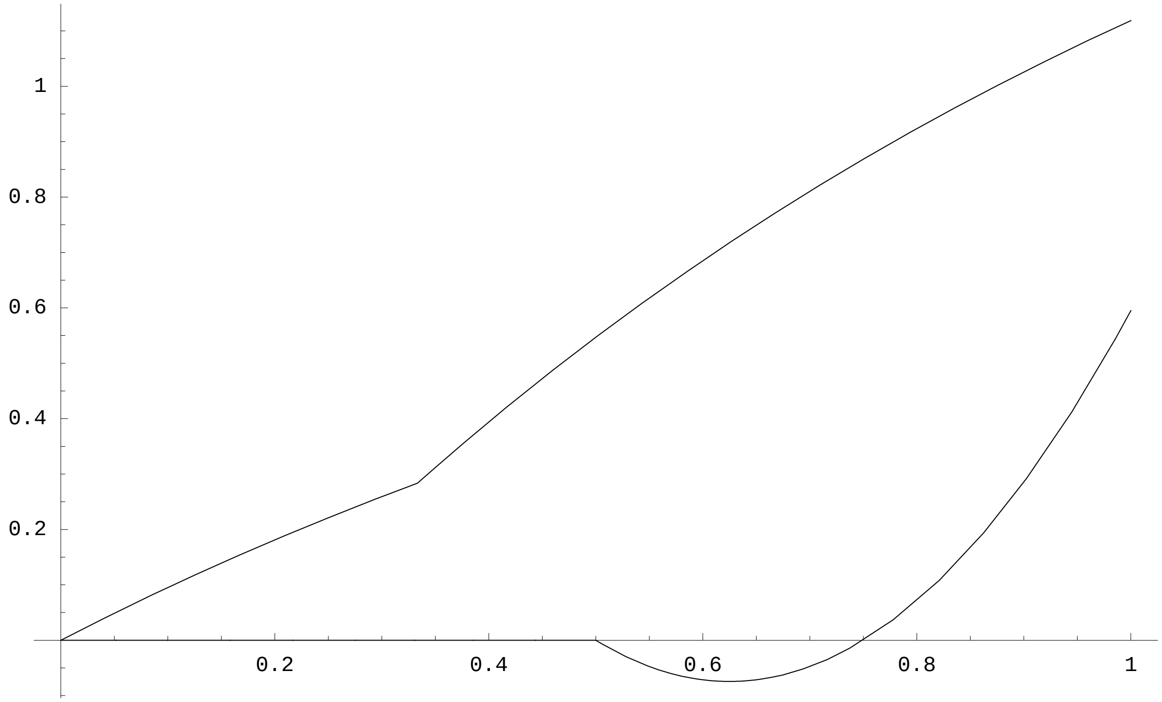
Rate $= 1.5$

Adaptive refinements, i.e. **awgm**

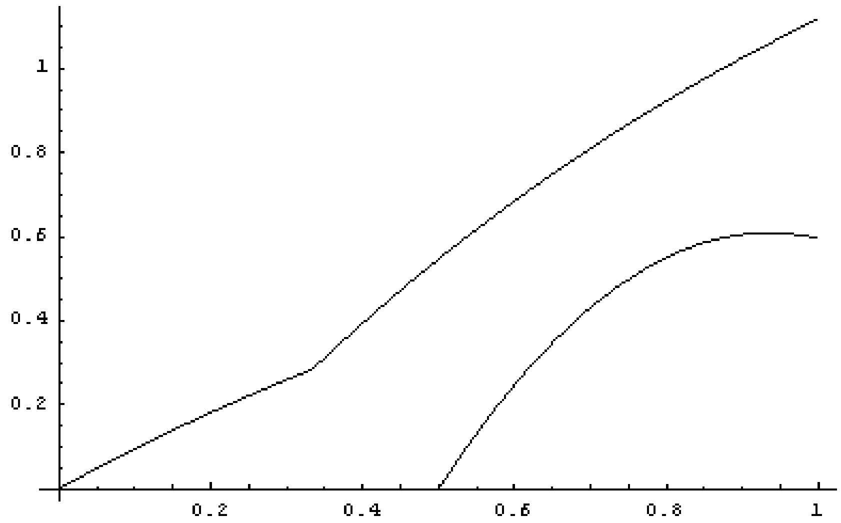


Rate $= 3$

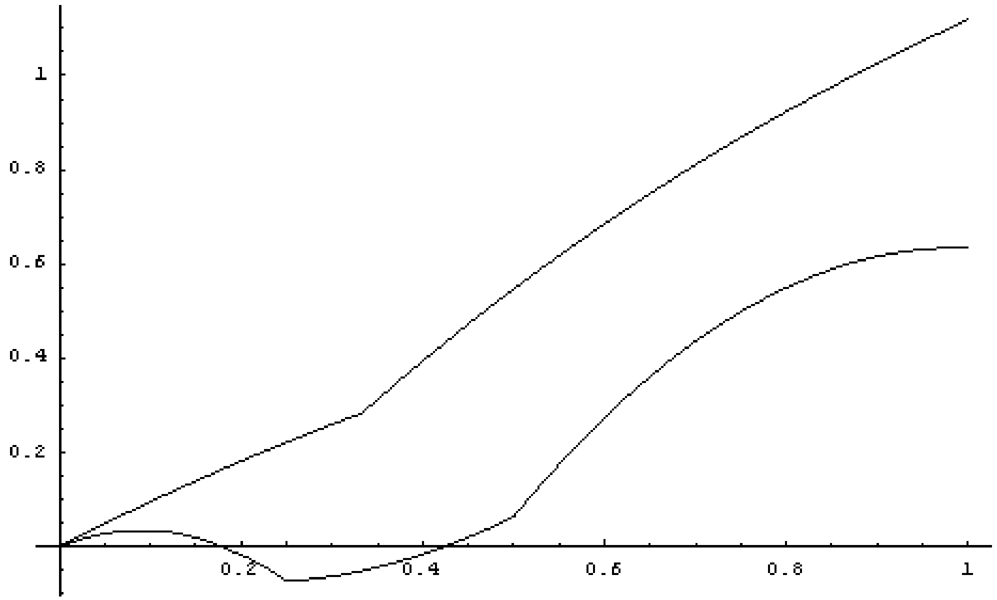
Some approximations



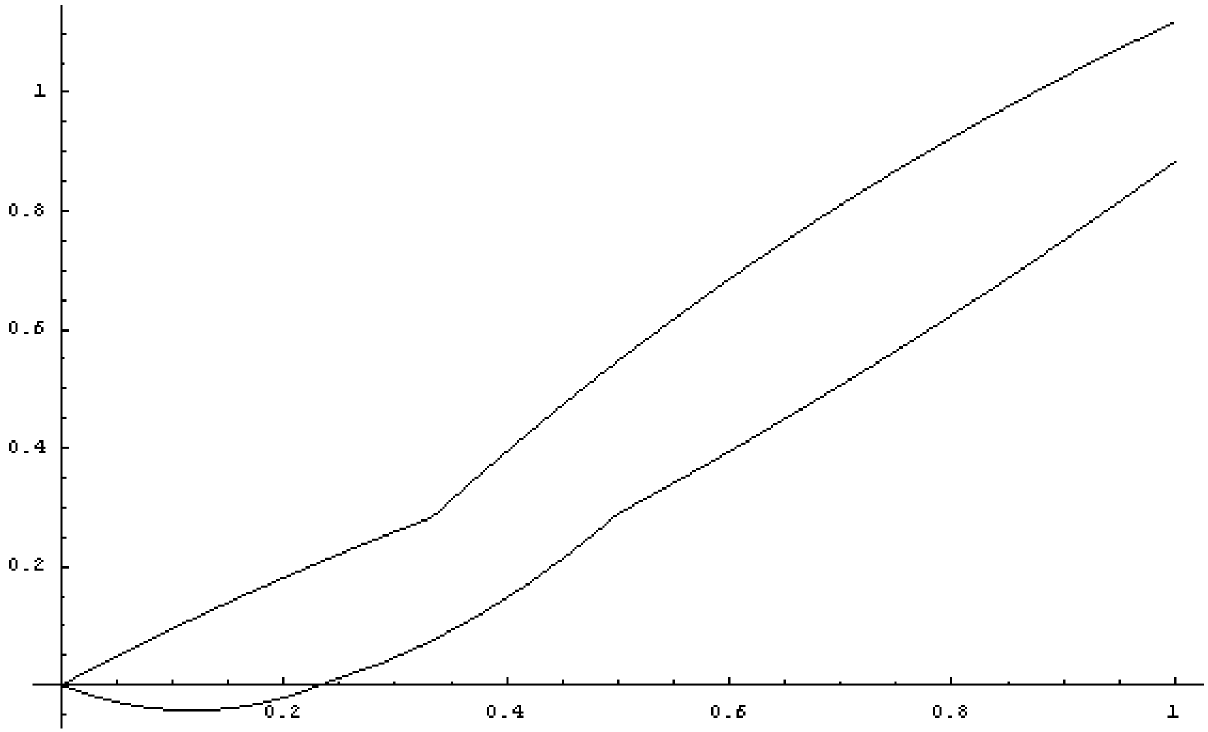
1 wavelet



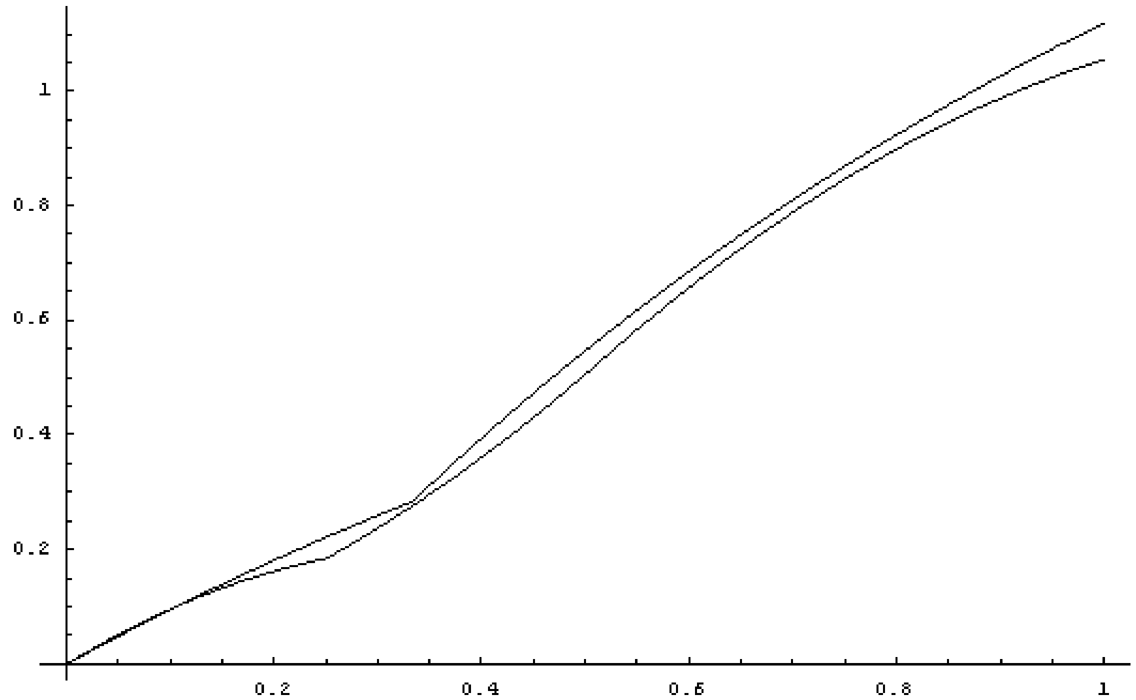
2 wavelets



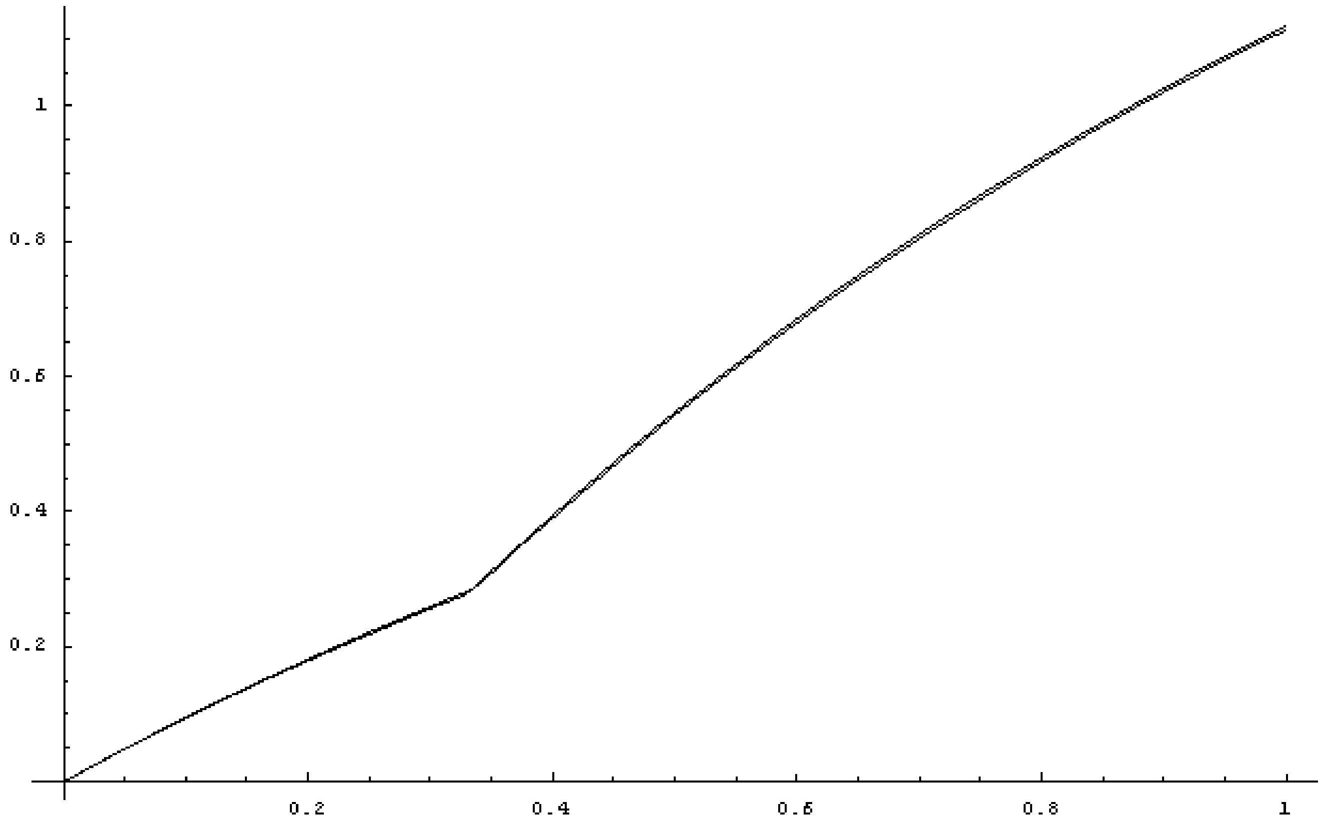
3 wavelets



4 wavelets



5 wavelets (all scaling functions are now in)



15 wavelets

Results for wavelets of order 5:

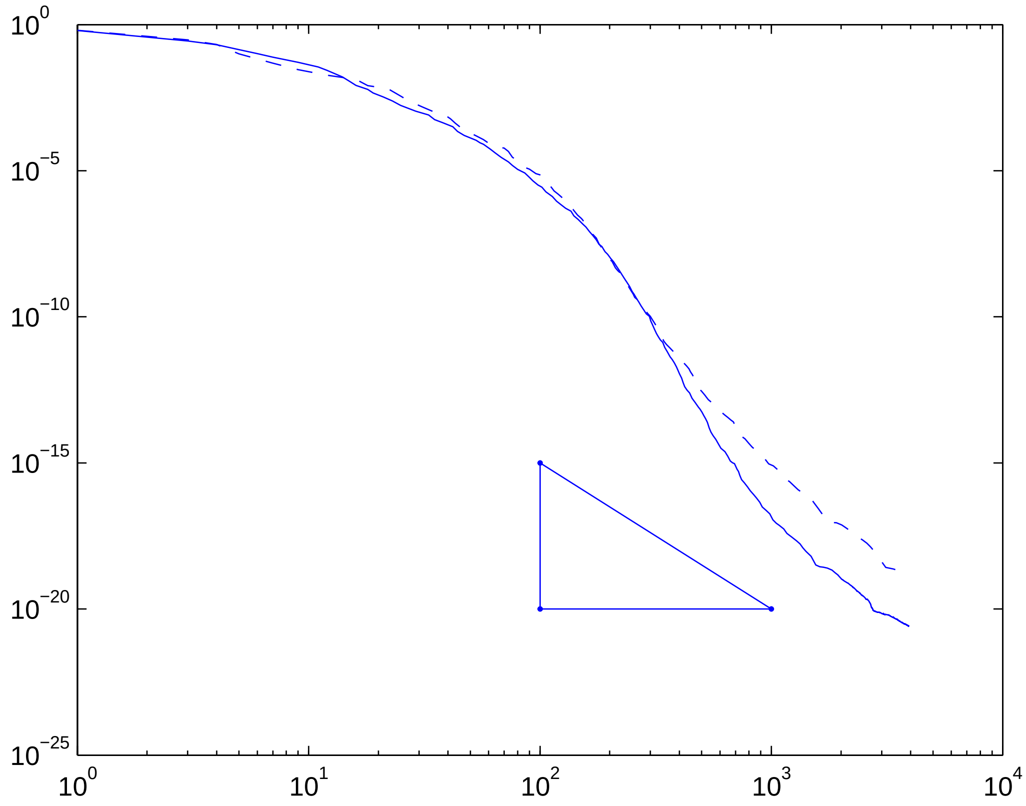


Figure 2: $\|\mathbf{B}\mathbf{u}_\varepsilon - \mathbf{f}\|/\|\mathbf{f}\|$ vs. $\#\mathrm{supp}\,\mathbf{u}_\varepsilon$ for $u_0 = 1$ (solid lines) and $u_0 = 0$ (dashed lines).

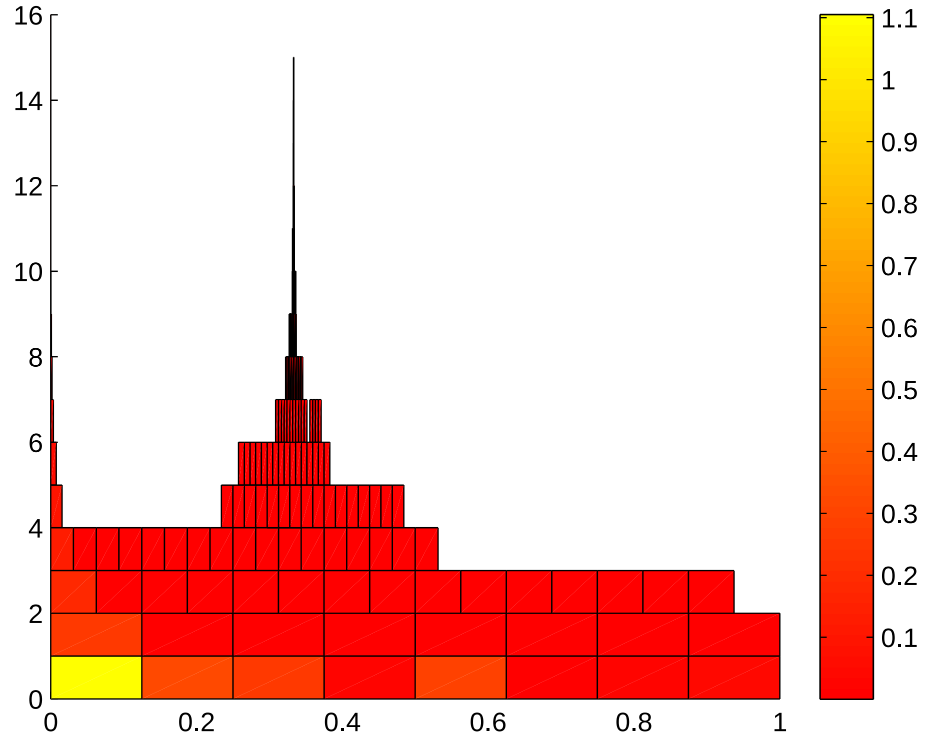


Figure 3: For $u_0 = 1$ and $\#\mathbf{u}_\varepsilon = 202$, the non-zero coefficients of $\mathbf{u}_\varepsilon$.

Numerical results heat eqn

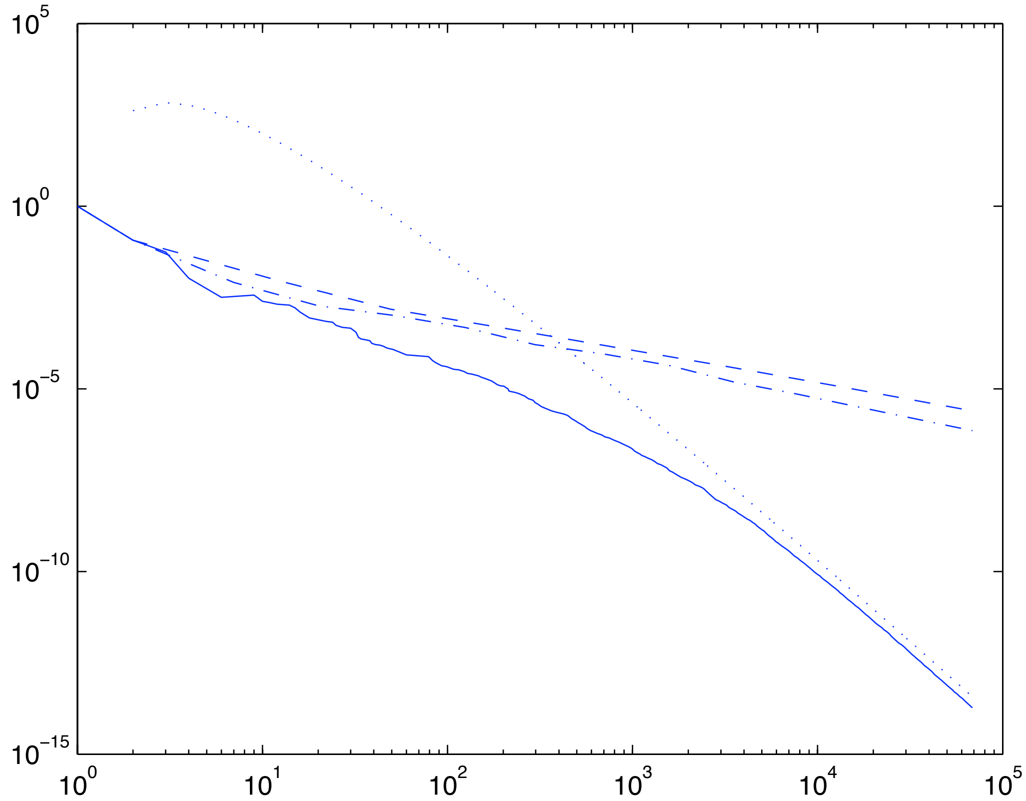


Figure 4: Heat eqn. in $n = 1$ spatial dimension, right-hand side $g = 1$ and initial condition $u_0 = 0$. $\|\mathbf{B}\mathbf{u}_\varepsilon - \mathbf{f}\|/\|\mathbf{f}\|$ vs. $N = \#\mathrm{supp}\,\mathbf{u}_\varepsilon$ for the **awgm** (solid), full-grid (dashed) and sparse-grid method (dashed-dotted). The dotted line is a multiple of $N^{-5}(\log N)^{5\frac{1}{2}}$.

Numerical results heat eqn

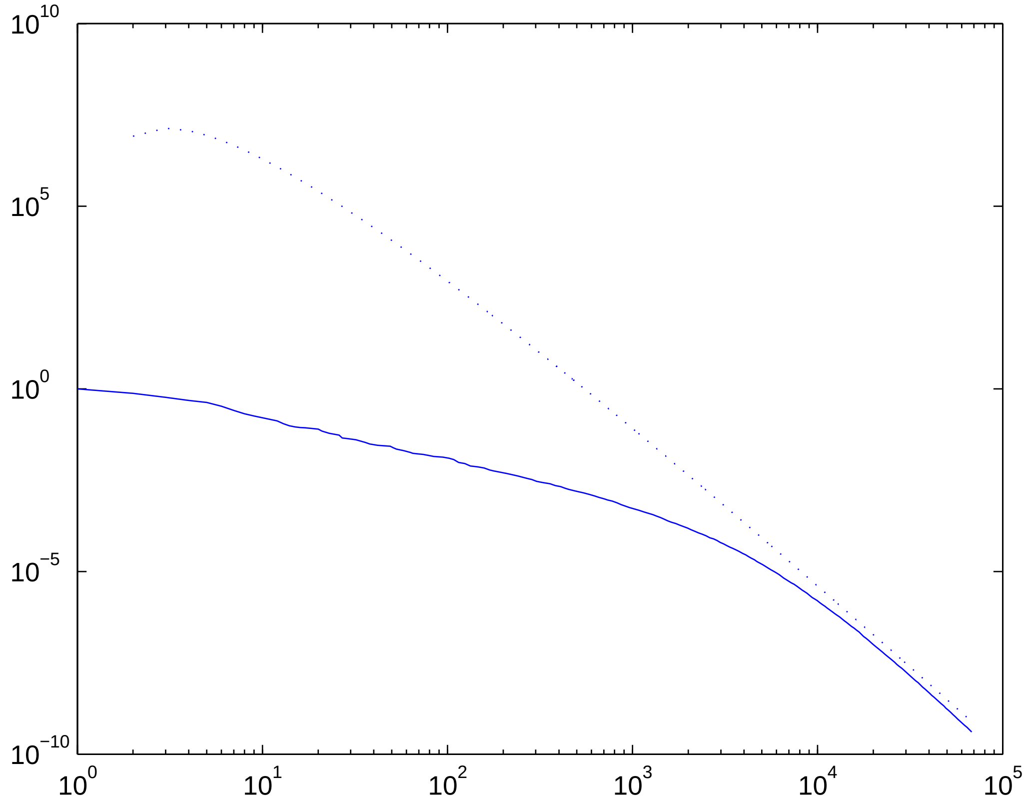


Figure 5: Heat eqn. in $n = 1$ spatial dimension, right-hand side $g = 1$ and initial condition $u_0 = 1$. $\|\mathbf{B}\mathbf{u}_\varepsilon - \mathbf{f}\|/\|\mathbf{f}\|$ vs. $N = \#\mathrm{supp}\,\mathbf{u}_\varepsilon$ for the **awgm** (solid). The dotted line is a multiple of $N^{-5}(\log N)^{5\frac{1}{2}}$.

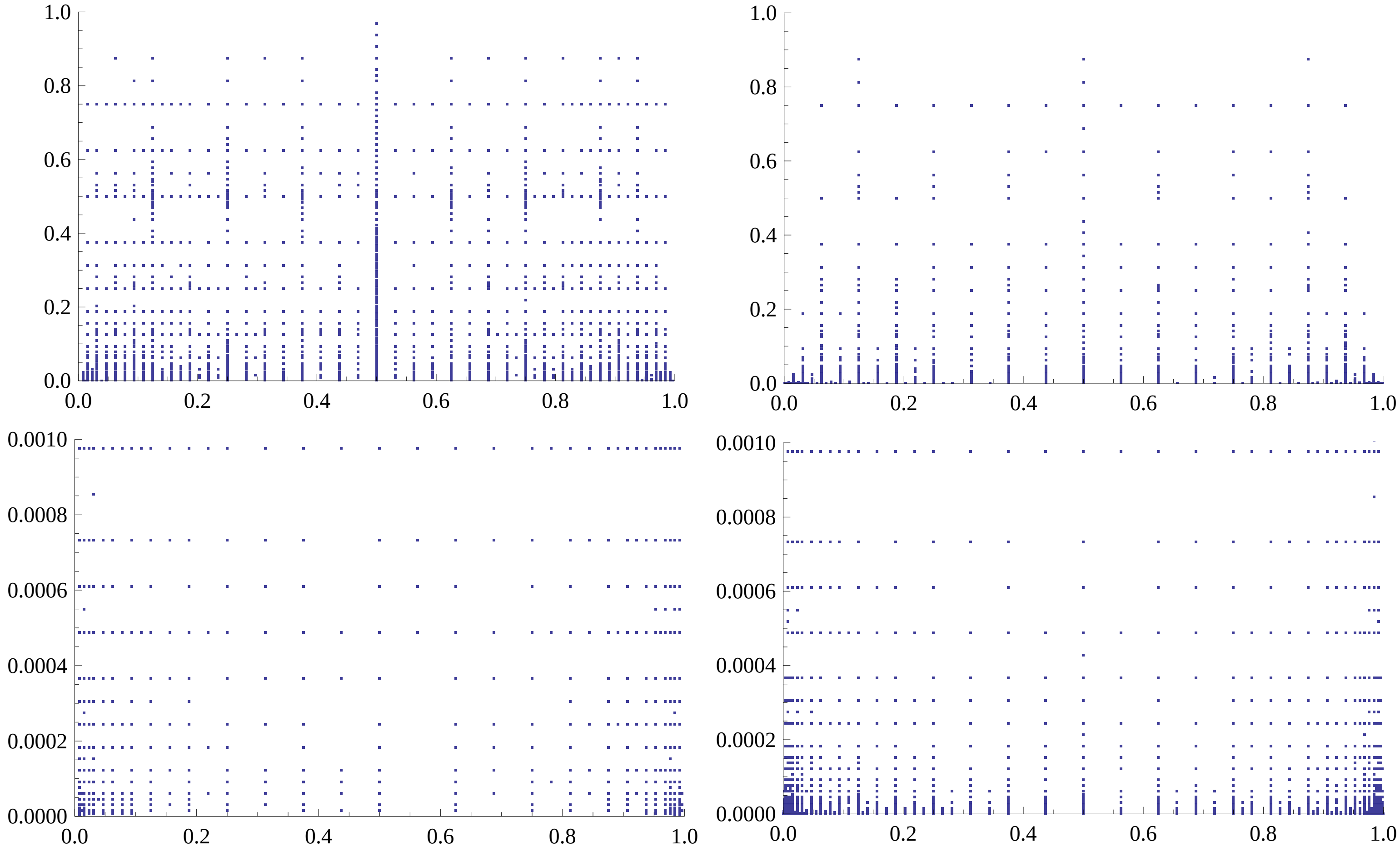


Figure 6: Heat eqn. in $n = 1$ spatial dimension and right-hand side $g = 1$. Centers of the supports of the wavelets selected by the **awgm**. Left $u_0 = 0$ and $\#\mathbf{u}_\varepsilon = 13420$. Right $u_0 = 1$ and $\#\mathbf{u}_\varepsilon = 13917$. A zoom in near $t = 0$ is given at the bottom row.

instat. (N)SE

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \mathbf{\Delta}_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \mathbf{g} & \text{on } I \times \Omega, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} = h & \text{on } I \times \Omega, \\ \mathbf{u} = 0 & \text{on } I \times \partial\Omega, \\ \mathbf{u}(0, \cdot) = 0 & \text{on } \Omega, \\ \int_\Omega p \, d\mathbf{x} = 0. \end{cases} \tag{1}$$

Can be reduced, for $h = 0$, to parabolic for velocities, but then arising spaces will be spaces of divergence-free functions. We enforce incompressibility constraint via Lagrange multiplier. Saddle point form.

Space-time variational form: With

$$\begin{cases} c(\mathbf{u}, \mathbf{v}) & := & \int_I \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} + \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v} \, d\mathbf{x} dt, \\ d(p, \mathbf{v}) & := & -\int_I \int_\Omega p \operatorname{div}_{\mathbf{x}} \mathbf{v} \, d\mathbf{x} dt, \\ f(\mathbf{v}, q) & := & \int_I \int_\Omega \mathbf{g} \cdot \mathbf{v} + h \, q \, d\mathbf{x} dt, \end{cases}$$

find $(\mathbf{u}, p)$ in some suitable space, such that

$$(\mathrm{S}(\mathbf{u}, p))(\mathbf{v}, q) := c(\mathbf{u}, \mathbf{v}) + d(p, \mathbf{v}) + d(q, \mathbf{u}) = f(\mathbf{v}, q)$$

for all $(\mathbf{v}, q)$ from another suitable space.

For $\delta \in \{0, T\}$,

$$\begin{aligned}
\breve{H}^s_{0,\{\delta\}}(I) &:= [L_2(I), H^1_{0,\{\delta\}}(I)]_s, \\
\hat{H}^s(\Omega) &:= [L_2(\Omega), H^2(\Omega) \cap H^1_0(\Omega)]_{\frac{s}{2}}, \\
\bar{H}^s(\Omega) &:= [(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s+1}{2}}, \\
\mathscr{U}^s_\delta &:= L_2(I; \hat{H}^{2s}(\Omega)^n) \cap \breve{H}^s_{0,\{\delta\}}(I; L_2(\Omega)^n), \\
\mathscr{P}^s_\delta &:= \big(L_2(I; \bar{H}^{2s-1}(\Omega)') \cap \breve{H}^{1-s}_{0,\{\delta\}}(I; \bar{H}^1(\Omega)')\big)'.
\end{aligned}$$

Thm 2 ([SS15]). *For $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain, and $s \in (\frac{1}{4}, \frac{3}{4})$, it holds that*

$$\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}^s_0 \times \mathscr{P}^s_T, (\mathscr{U}^{1-s}_T \times \mathscr{P}^{1-s}_0)').$$

- For $\partial\Omega \in C^2$, result also valid for $s \in [0, 1]$. $s \in \{0, 1\}$ avoids fractional Sob. sp., but $\mathscr{U}^1_\delta$ involves $H^2(\Omega)$.
- For $s \in (\frac{1}{4}, \frac{3}{4})$, all arising spaces can be 'conveniently' equipped with wavelet Riesz bases. **awgm** applies.
- Generalizes to NSE for $n = 2$; for $n = 3$ we need 's' $> \frac{3}{4}$ which requires smooth $\partial\Omega$ or convex domains, and C^1-wavelets.

A few ingredients of proof of Thm 2:

- two inf-sup conditions proven using right-inverse div^+ of div constructed in [Bog79].
- **maximal regularity** of evolution operators used to show bounded invertibility of the parabolic problem on the divergence-free velocities.
- With $(A\mathbf{u})(\mathbf{v}) := \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, d\mathbf{x}$ on $\hat{H}^1(\mathrm{div}\, 0; \Omega) \times \hat{H}^1(\mathrm{div}\, 0; \Omega)$, **elliptic regularity** on Lipschitz domains gives that for $\varsigma \in [0, \frac{3}{4})$, $\hat{H}^{2\varsigma}(\mathrm{div}\, 0; \Omega) \simeq [\hat{H}^0(\mathrm{div}\, 0; \Omega), D(A)]_\varsigma$.

Conclusions

- Adaptive wavelet method solves general well-posed operator equations with the best possible rate in linear complexity.
- Main advantage is that a posteriori estimator, being the residual of operator equation in wavelet coordinates, is not restricted to elliptic problems.
- Numerical results for nontrivial problems needed.
- Well-posedness of space-time variational formulations of evolution problems is not only of interest for wavelet methods.

Thanks for your attention!

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