On nonlinear elliptic functional equations

L. Simon

Institute of Mathematics
L. Eötvös University of Budapest

FSDONA 2016, Prague

The main topics of this talk

In the present talk we shall consider weak solutions of the following boundary value problems for elliptic functional differential equations:

$$-\sum_{j=1}^{n} D_j[a_j(x, u, Du; u)] + a_0(x, u, Du; u) = F(x), \quad x \in \Omega \tag{1}$$

$$\gamma(u(x); u) = \varphi(x), \quad x \in \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a (possibly unbounded) domain and ; u denotes nonlocal dependence on u.
By using the theory of monotone type operators my PhD student M. Csirik proved an existence theorem for $2m$ order nonlinear elliptic functional differential equations. After formulating the existence theorem, we shall investigate the number of solutions in certain particular cases.

Denote by $\Omega \subset \mathbb{R}^n$ a (possibly unbounded) domain, $1 < p < \infty$, $W^{1,p}(\Omega)$ the Sobolev space with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^p + |u|^p \right) dx \right]^{1/p}.$$

Further, let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace of $W^{1,p}(\Omega)$, $V^\star$ the dual space of V, the duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$. Now we formulate the assumptions of the existence theorem for second order equations.

(A_1). The functions $a_j : \Omega \times \mathbb{R}^{n+1} \times V \to \mathbb{R}$ $(j = 0, 1, \cdots, n)$ satisfy the Carathéodory conditions for arbitrary fixed $u \in V$.

(A_2). There exist bounded (nonlinear) operators $g_1 : V \to \mathbb{R}^+$ and $k_1 : V \to L^q(\Omega)$ $(1/p + 1/q = 1)$ such that k_1 is compact and

$$|a_j(x, \eta, \zeta; u)| \leq g_1(u)[1 + \eta|^{p-1} + |\zeta|^{p-1}] + [k_1(u)](x),$$

$j = 0, 1, \cdots, n$, for a.e. $x \in \Omega$, each $(\eta, \zeta) \in \mathbb{R}^{n+1}$, $u \in V$.

(A_3). The inequality

$$\sum_{j=1}^{n} [a_j(x, \eta, \zeta; u) - [a_j(x, \eta, \zeta^\star; u)](\zeta_j - \zeta_j^\star) \geq g_2(u)|\zeta - \zeta^\star|^p$$

holds where $g_2(u) \geq c^\star(1 + \|u\|_V)^{-\sigma^\star}$ and the constants $c^\star, \sigma^\star$ satisfy $c^\star > 0$, $0 \leq \sigma^\star < p - 1$.

$(A4)$ The inequality

$$\sum_{j=0}^{n} a_j(x, \eta, \zeta; u)\xi_j \geq g_2(u)[1 + |\eta|^p + |\zeta|^p] - [k_2(u)](x)$$

holds where $\xi = (\eta, \zeta)$, the operator $k_2 : V \to L^1(\Omega)$ satisfies

$$\|k_2(u)\|_{L^1(\Omega)} \leq \text{const}(1 + \|u\|_V)^\sigma, \quad u \in V$$

with some positive $\sigma < p - \sigma^\star$.
$(A5)$ If $(u_k) \to u$ weakly in V, $(\eta^k) \to \eta$ in $\mathbb{R}$, $(\zeta^k) \to \zeta$ in $\mathbb{R}^n$ then for a subsequence, a.a. $x \in \Omega$

$$\lim_{k\to\infty} a_j(x, \eta^k, \zeta^k; u_k) = a_j(x, \eta, \zeta; u), \quad j = 0, 1, \cdots, n.$$

Theorem

Assume (A_1) - (A_5). Then the operator $A : V \to V^\star$ defined by

$$\langle A(u), v \rangle = \int_\Omega \left[\sum_{j=1}^{n} a_j(x, u, Du; u) D_j v + a_0(x, u, Du; u) v \right] dx$$

is bounded, pseudomonotone and coercive. Thus for any $F \in V^\star$ there exists $u \in V$ satisfying $A(u) = F$. (M. Csirik, EJQTDE, 2016.)

Main steps of the proof Assumptions (A_1), (A_2) directly imply that A is bounded and (A_4) implies that A is coercive. The proof of pseudomonotonicity is not difficult if Ω is bounded (since $W^{1,p}(\Omega)$ is compactly imbedded in $L^p(\Omega)$). If Ω is unbounded, one can use arguments of F. E Browder. (Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains, Proc. Nat. Ac. Sci. 74, 2659-2661.)

When Ω is bounded (with sufficiently smooth boundary):

$$a_j(x,\eta,\zeta;u) = b(x,[H(u)](x)\zeta_j|\zeta|^{p-2}, \quad j = 1,\cdots,n,$$

$$a_0(x,\eta,\zeta;u) = b_0(x,[H_0(u)](x))\eta|\eta|^{p-2}+\hat{b}_0(x,[F_0(u)](x))\hat{\alpha}_0(x,\eta,\zeta)$$

where $b, b_0, \hat{b}_0, \hat{\alpha}_0$ are Carathéodory functions satisfying

$$b(x,\theta), \quad b_0(x,\theta) \geq \frac{c_2}{1+|\theta|^{\sigma^\star}}, \quad (c_2 > 0, \quad 0 \leq \sigma^\star < p-1)$$

$$|\hat{b}_0(x,\theta)| \leq 1 + |\theta|^{p-1-\rho^\star}, \quad (0 < \rho^\star < p-1)$$

$$|\hat{\alpha}_0(x,\eta,\zeta)| \leq c_1[1+|\eta|^{\hat{\rho}} + |\zeta|^{\hat{\rho}}, \quad (0 \leq \hat{\rho}, \quad \sigma^\star + \hat{\rho} < \rho^\star;$$

$$H, H_0 : L^p(\Omega) \to C(\overline{\Omega}), \quad F_0 : L^p(\Omega) \to L^p(\Omega)$$

are linear continuous operators. If b, b_0 are between two positive constants then $H, H_0 : L^p(\Omega) \to L^p(\Omega)$ is admitted (e.g. u with transformed argument).

In the case when Ω is unbounded, the above functions a_j satisfy the assumptions of the existence theorem if

$$H, H_0 : L^p(\Omega') \to C(\overline{\Omega})$$

are bounded linear operators with some bounded domain Ω', further, $\hat{\alpha}_0 = 1$ and $\hat{b}_0$ has the form

$$\hat{b}_0(x; u) = b_1(x)N(u)$$

where

$$N : V \to W^{1,p}(\Omega) \text{ or } N : V \to \mathbb{R}$$

is a bounded linear operator and

$$b_1 \in L^s(\Omega) \text{ where } \frac{p}{p-2+p/n} < s < \frac{p}{p-2}.$$

Now consider particular cases for the functions a_j, g:

$$a_j(x, \eta, \zeta; u) = \tilde{a}_j(x, \eta, \zeta, M(u)), \quad \gamma(u; u) = \tilde{\gamma}(u, M(u))$$

$j = 0, 1, \cdots, n$, (first boundary condition, for simplicity), where $M : V \to \mathbb{R}$ is a bounded, continuous (possibly nonlinear) operator and

$$\tilde{a}_j : \Omega \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}, \quad \tilde{\gamma} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$$

satisfy the Carathéodory conditions.
Assume that for every $\lambda \in \mathbb{R}$ there exists a unique solution $u_\lambda \in V$ of

$$A_\lambda(u_\lambda) = F \quad (F \in V^\star), \tilde{\gamma}(u, \lambda) = \varphi \text{ on } \partial\Omega$$

where $A_\lambda : V \to V^\star$ is defined by

$$\langle A_\lambda(u), v \rangle = \int_\Omega \left[\sum_{j=1}^{n} \tilde{a}_j(x, u, Du, \lambda) D_j v + \tilde{a}_0(x, u, Du, \lambda) v \right] dx$$

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(\lambda) = M(u_\lambda)$. Then a function $u \in V$ is a solution of

$$\int_\Omega \left[\sum_{j=1}^n \tilde{a}_j(x, u, Du, M(u)) D_j v + \tilde{a}_0(x, u, Du, M(u)) v \right] dx = \quad (3)$$

$$\langle F, v \rangle, \quad \tilde{\gamma}(u, M(u)) = \varphi \text{ on } \partial\Omega$$

if and only if $\lambda = M(u)$ satisfies $\lambda = g(\lambda)$.
Consider the following particular case

$$\tilde{a}_j(x, u, Du, M(u)) = b_j(x, u, Du) h(M(u)), \text{ i.e.}$$

$$\tilde{a}_j(x, u, Du, \lambda) = b_j(x, u, Du) h(\lambda),$$

$j = 1, \cdots, n$ and

$$\tilde{a}_0(x, u, Du, \lambda) = b_0(x, u, Du) h(\lambda) + \beta(x) l(\lambda),$$

$$\tilde{\gamma}(u,\lambda) = h(\lambda)u + \beta_1(x)l_1(\lambda)$$

with some continuous functions $h : \mathbb{R} \to \mathbb{R}^+$, $l, l_1 : \mathbb{R} \to \mathbb{R}$ and $\beta \in L^q(\Omega)$, $\beta_1 \in L^p(\partial\Omega)$. Then

$$A_\lambda(u) = F, \quad \tilde{\gamma}(u, M(u)) = \varphi \text{ on } \partial\Omega$$

can be written in the form

$$B(u) = \frac{F - l(\lambda)\beta}{h(\lambda)}, \quad u = \frac{\varphi - l_1(\lambda)\beta_1}{h(\lambda)} \text{ on } \partial\Omega$$

where $B(u)$ is defined by

$$\langle B(u), v\rangle = \int_\Omega \left[\sum_{j=1}^n b_j(x,u,Du)D_j v + b_0(x,u,Du)v\right],$$

$$u \in W^{1,p}(\Omega), \quad v \in V.$$

Assume that $B : V \to V^\star$ is a uniformly monotone, bounded, hemicontinuous operator then the unique solution of

$$A_\lambda(u) = F, \quad \tilde{\gamma}(u, M(u)) = \varphi \text{ on } \partial\Omega :$$

$$u = u_\lambda = \mathcal{B}^{-1}\left(\frac{F - l(\lambda)\beta}{h(\lambda)}, \frac{\varphi - l_1(\lambda)\beta_1}{h(\lambda)}\right)$$

where $\mathcal{B}(u) = (B(u), u\mid_{\partial\Omega})$ and thus

$$g(\lambda) = M(u_\lambda) = M\left[\mathcal{B}^{-1}\left(\frac{F - l(\lambda)\beta}{h(\lambda)}, \frac{\varphi - l_1(\lambda)\beta_1}{h(\lambda)}\right)\right].$$

Since $\mathcal{B}^{-1} : V^\star \times L^p(\partial\Omega) \to V$ and $M : V \to \mathbb{R}$, l, h are continuous, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Further, we have shown that the number of solutions of problem (1), (2) equals to the number of real solutions of the equation $g(\lambda) = \lambda$.

Now consider two particular cases.

1. Assume that $\varphi = 0$ and B, M are homogeneous in the sense

$$B^{-1}(\mu F) = \mu^{\frac{1}{p-1}} B^{-1}(F) \text{ for all } \mu \geq 0 \quad (p > 1),$$

$$M(\mu u) = \mu^{\sigma} M(u) \text{ for all } \mu \geq 0 \quad (\sigma \geq 0)$$

(M is nonnegative). Then

$$g(\lambda) = \frac{M\{B^{-1}[F - l(\lambda)\beta]\}}{h(\lambda)^{\frac{\sigma}{p-1}}}.$$

Consequently, if g is a positive continuous function such that $\lambda = g(\lambda)$ has exactly N roots ($N = 0, 1, \cdots, \infty$) then our boundary value problem (with 0 boundary condition) has exactly N solutions with

$$h(\lambda) = \left[\frac{M\{B^{-1}[F - l(\lambda)\beta]\}}{g(\lambda)}\right]^{\frac{p-1}{\sigma}}.$$

We have this particular case if e.g. B is defined by the p-Laplacian, i.e.

$$b_j(x,\eta,\zeta) = |\zeta|^{p-2}\zeta, \quad j = 1,\cdots,n, \quad b_0(x,\eta,\zeta) = c|\eta|^{p-2}\eta$$

$\eta \in \mathbb{R}$, $\zeta \in \mathbb{R}^n$ with some $c > 0$. (If Ω is bounded then c may be 0, too.) Further,

$$M(u) = \int_\Omega \left[\sum_{j=1}^n a_j(x)|D_j u|^\sigma + a_0(x)|u|^\sigma \right] dx$$

where $a_j \in L^\infty(\Omega)$, $a_j > 0$, $0 < \sigma \le p$.

2. Assume that B and M are linear Then

$$g(\lambda) = \frac{M[\mathcal{B}^{-1}(F,\varphi)] - l(\lambda)M[\mathcal{B}^{-1}(\beta,0)] - l_1(\lambda)M[\mathcal{B}^{-1}(0,\beta_1)]}{h(\lambda)}.$$

Therefore, if g is a positive continuous function such that $\lambda = g(\lambda)$ has N roots ($N = 0, 1, \cdots, \infty$) then our boundary value problem has N solutions with

$$h(\lambda) = \frac{M[\mathcal{B}^{-1}(F,\varphi)] - l(\lambda)M[\mathcal{B}^{-1}(\beta,0)] - l_1(\lambda)M[\mathcal{B}^{-1}(0,\beta_1)]}{g(\lambda)}$$

and arbitrary continuous functions l, l_1. Similarly, if $M[\mathcal{B}^{-1}(\beta,0)] \neq 0$ and g is a continuous function such that $\lambda = g(\lambda)$ has N roots then our boundary value problem has N solutions with

$$l(\lambda) = \frac{g(\lambda)h(\lambda) - M[\mathcal{B}^{-1}(F,\varphi)] + l_1(\lambda)M[\mathcal{B}^{-1}(0,\beta_1)]}{M[\mathcal{B}^{-1}(\beta,0)]}$$

and arbitrary continuous functions h, l_1.

In this case operator $M : W^{1,2}(\Omega) \to \mathbb{R}$ may have the form

$$Mu = \int \left[\sum_{j=1}^{n} a_j D_j u + a_0 u \right] + \int_{\partial\Omega} b_0 u d\sigma$$

where $a_j \in L^2(\Omega)$, $b_0 \in L^2(\partial\Omega)$.

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