Interpolation of Morrey spaces and Related Smoothness Spaces

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1. Introduction - Morrey spaces

- Caccioppoli inequalities;
- reverse Hölder inequality

$$|B|^{-1}\int_B |f(x)|^q dx \le c\left(|B|^{-1}\int_B |f(x)| dx\right)^q;$$

- Muckenhoupt weights

$$|B|^{-1}\int_B |\omega(x)|\, dx \le c\left(|B|^{-1}\int_B |\omega(x)|^{-p'/p} dx\right)^{p/p'}.$$

$0 < p \le u \le \infty$:

$f \in L_p^{\ell oc}(\mathbb{R}^n)$ belongs to the Morrey space $\mathcal{M}_p^u(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{M}_p^u(\mathbb{R}^n)} := \sup_{r>0} \sup_{y\in\mathbb{R}^n} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx \Big)^{1/p}$$

- Morrey (1938)
- $\mathcal{M}_p^p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\mathcal{M}_p^\infty(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.
- Kozono, Yamazaki (1994): *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*.

- $\|f(\lambda\,\cdot\,)\|_{\mathcal{M}^u_p(\mathbb{R}^n)} = \lambda^{-n/u}\,\|f\|_{\mathcal{M}^u_p(\mathbb{R}^n)}, \qquad \lambda > 0.$

- Extremal functions: $f_\alpha(x) := |x|^{-\alpha}$, $x \neq 0$, $\alpha > 0$.

$$f_\alpha \in \mathcal{M}^u_p(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad \alpha = \frac{n}{u} \quad \text{and} \quad \alpha < \frac{n}{p}\,.$$

- $\mathcal{M}^u_p(\mathbb{R}^n)$ is not separable $\quad \Longleftrightarrow \quad p \neq u$

$$\Longleftrightarrow \quad \mathcal{M}^u_p(\mathbb{R}^n) \neq L_p(\mathbb{R}^n).$$

2. Some interpolation methods

- Real interpolation;
- complex interpolation;
- the $\pm$-method of Gustavsson and Peetre;
- the Calderón product;
- a second complex interpolation method by Calderón.

2.1 Real interpolation

For any $t \in (0, \infty)$ and any $x \in X_0 + X_1$, define

$$K(t, x, X_0, X_1) := \inf_{\substack{x = x_0 + x_1 \\ x_0 \in X_0,\, x_1 \in X_1}} \{ \|x_0\|_{X_0} + t \, \| \, x_1 \, \|_{X_1} \} \, .$$

Let $\Theta \in (0, 1)$ and $q \in (0, \infty]$. The real interpolation space $(X_0, X_1)_{\Theta, q}$ is defined as the collection of all $x \in X_0 + X_1$ such that

$$\| \, x \, \|_{(X_0, X_1)_{\Theta,q}} := \left\{ \int_0^\infty [t^{-\Theta} \, K(t, x, X_0, X_1)]^q \, \frac{dt}{t} \right\}^{1/q} < \infty \, .$$

- Lions/Peetre 1958-1964.

$0 < p_0, p_1 \le \infty$:

$$(L_{p_0}, L_{p_1})_{\Theta, p} = L_p \, , \qquad \frac{1}{p} = \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1} \, .$$

2.2 Complex interpolation

- (X_0, X_1) interpolation couple of quasi-Banach spaces;
- $X_0 + X_1$ analytically convex;
- $S_0 := \{z : \quad 0 < \Re e\, z < 1\}, \quad S := \overline{S_0}$;

$\mathcal{A} := \mathcal{A}(X_0, X_1)$ - set of all bounded and analytic functions $f : S_0 \to X_0 + X_1$, which extend continuously to the closure S of the strip S_0 s. t. the traces $t \mapsto f(j + it)$ are bounded continuous functions into X_j, $j \in \{0, 1\}$. We endow $\mathcal{A}$ with the quasi-norm

$$\| f \|_{\mathcal{A}} := \max \left\{ \sup_{t \in \mathbb{R}} \| f(it) \|_{X_0}, \ \sup_{t \in \mathbb{R}} \| f(1+it) \|_{X_1} \right\} .$$

The *complex interpolation space* $[X_0, X_1]_\Theta$ is defined as the set of all $x \in \mathcal{A}(\Theta) := \{f(\Theta) : \; f \in \mathcal{A}\}$.
We put

$$\| \, x \, \|_{[X_0,X_1]_\Theta} := \inf \Big\{ \| \, f \, \|_{\mathcal{A}} : \; f \in \mathcal{A}, \; f(\Theta) = x \Big\} \, .$$

- Calderon (1963/64), Lions (1960), Krein (1960).
- Kalton (1986), Kalton, Mayboroda, Mitrea (2007).

$0 < p_0, p_1 < \infty$:

$$[L_{p_0}, L_{p_1}]_\Theta = L_p \, , \qquad \frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \, .$$

2.3 The second complex method of interpolation

Let (X_0, X_1) be an interpolation couple of Banach spaces, i. e., X_0 and X_1 are continuously embedded into a larger topological vector space Y. Let $\Theta \in (0,1)$.
Let $\mathcal{G} := \mathcal{G}(X_0, X_1)$ be the set of all functions $f : S \to X_0 + X_1$ such that

(a) $\frac{f(\cdot)}{1+|\cdot|}$ is continuous and bounded on S;

(b) f is analytic in S_0;

(c) $f(j + it_1) - f(j + it_2)$ has values in X_j for all $(t_1, t_2) \in \mathbb{R}^2$, $j \in \{0, 1\}$;

(d) the quantity

$$\| f \|_{\mathcal{G}} := \max \Big\{ \sup_{t_1 \neq t_2} \Big\| \frac{f(it_2) - f(it_1)}{t_2 - t_1} \Big\|_{X_0} , \sup_{t_1 \neq t_2} \Big\| \frac{f(1+it_2) - f(1+it_1)}{t_2 - t_1} \Big\|_{X_1} \Big\}$$

is finite.

The complex interpolation space $[X_0, X_1]^\Theta$ is defined as the set of all $x \in \mathcal{G}(\Theta) := \{f(\Theta) : \ f \in \mathcal{G}\}$ and, for all $x \in \mathcal{G}(\Theta)$,

$$\| x \|_{[X_0,X_1]^\Theta} := \inf \Big\{ \| f \|_{\mathcal{G}} : \ f \in \mathcal{G}, \ f(\Theta) = x \Big\} .$$

3. Interpolation of Morrey spaces

- Stampacchia (1964), Campanato, Murthy (1965), Spanne (1966), Peetre (1969);
- Cobos, Peetre, Persson (1998);
- Ruiz, Vega (1995), Blasco, Ruiz, Vega (1999) - some negative results in specialized situations;
- Lemarié-Rieusset (2013), Lemarié-Rieusset (2014);
- Yang, Yuan, Zhuo (2013), Lu, Yang, Yuan (2014); Yuan, S., Yang (2015);
- Hakim, Sawano (2016); Hakim, Nakamura, Sawano (2016);
- Burenkov, Nursultanov (2010), Burenkov, Chigambayeva, Nursultanov (2014,2016);
- Interpolation of smoothness spaces built on Morrey spaces.

3.1 Interpolation of Morrey spaces - the complex method

From now on we always assume

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \, . \qquad (1)$$

and

$$\frac{1}{u} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1} \, . \qquad (2)$$

Lemarié-Rieusset (2013):

$$[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\Theta \neq \mathcal{M}^{u}_{p}(\mathbb{R}^n) \quad \text{if} \quad p_0\, u_1 \neq p_1\, u_0 .$$

Cobos, Peetre, Persson (1998), Yuan, S., Yang (2014):

$$[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\Theta \hookrightarrow \mathcal{M}^{u}_{p}(\mathbb{R}^n)$$

and the embedding is always proper except the trivial cases

$$[\mathcal{M}^{p_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{p_1}(\mathbb{R}^n)]_\Theta \qquad \text{and} \qquad [\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n)]_\Theta\,.$$

3.2 Interpolation of Morrey spaces - the real method

Lemarié-Rieusset (2013), Yuan, S., Yang (2015):

$$(\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n))_{\Theta,q} \neq \mathcal{M}^{u}_{p}(\mathbb{R}^n) \qquad \text{for all } q\,,$$

except the trivial cases

(a) $p_0 = p_1$ and $u_0 = u_1$ or

(b) $p_0 = u_0$, $p_1 = u_1$ and $q = p$.

Local Morrey spaces

Burenkov, Nursultanov (2010): Local Morrey-type spaces

$$\|f\|_{LM^{\lambda}_{p,\infty}(\mathbb{R}^n)} := \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(0,r))} < \infty \, .$$

Let $0 < p \le \infty$, $0 < \Theta < 1$ and $0 \le \lambda_0, \lambda_1 \le n/p$. Then

$$\Big(LM^{\lambda_0}_{p,\infty}(\mathbb{R}^n), LM^{\lambda_1}_{p,\infty}(\mathbb{R}^n) \Big)_{\Theta,\infty} = LM^{\lambda}_{p,\infty}(\mathbb{R}^n) \, ,$$

where $\lambda := (1 - \Theta)\lambda_0 + \Theta_1 \lambda_1$.

3.3 Interpolation of Morrey spaces - positive results

$\mathring{\mathcal{M}}^{u_0}_{p_0}(\mathbb{R}^n)$ - closure of the test functions in $\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n)$.

Yang, Yuan, Zhuo (2013) proved

$$[\mathring{\mathcal{M}}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\Theta = [\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathring{\mathcal{M}}^{u_1}_{p_1}(\mathbb{R}^n)]_\Theta = \mathring{\mathcal{M}}^{u}_{p}(\mathbb{R}^n)$$

if

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\, , \qquad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$$

and

$$p_0\, u_1 = p_1\, u_0\, .$$

3.4 The Calderón product

$(\mathfrak{X}, \mathcal{S}, \mu)$: σ-finite measure space;
$\mathfrak{M}$ - the class of all complex-valued, μ-measurable functions on $\mathfrak{X}$;
$X_j \subset \mathfrak{M}$, $j \in \{0,1\}$ quasi-Banach lattices of functions, and $\Theta \in (0,1)$. Then the *Calderón product* $X_0^{1-\Theta} X_1^{\Theta}$ of X_0 and X_1 is the collection of all functions $f \in \mathfrak{M}$ such that the quasi-norm

$$\|f\|_{X_0^{1-\Theta}X_1^{\Theta}} := \inf\Big\{\|f_0\|_{X_0}^{1-\Theta}\|f_1\|_{X_1}^{\Theta} : |f| \le |f_0|^{1-\Theta}|f_1|^{\Theta} \quad \mu\text{-a.e.}, \; f_j \in X_j,\, j \in \{0,1\}\Big\}$$

is finite.

- Disadvantage: restricted to lattices; does not have the interpolation property.
- Morrey spaces are quasi-Banach lattices.
- Smoothness spaces, built on Morrey spaces, like Sobolev-Morrey spaces or Besov-Morrey spaces, are not lattices in general.
- Advantage: easy to calculate in many concrete situations.

Yang, Yuan, Zhuo (2013), Yuan, S., Yang (2015).

Theorem 1

Let $\Theta \in (0,1)$, $0 < p_0 \le u_0 < \infty$ *and* $0 < p_1 \le u_1 < \infty$ *such that* (1) *and* (2) *hold.*

(i) *It holds that*

$$\big[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n)\big]^{1-\Theta}\,\big[\mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)\big]^{\Theta} \hookrightarrow \mathcal{M}^{u}_{p}(\mathbb{R}^n)\,.$$

(ii) *If* $u_0\,p_1 = u_1\,p_0$, *then*

$$\big[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n)\big]^{1-\Theta}\,\big[\mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)\big]^{\Theta} = \mathcal{M}^{u}_{p}(\mathbb{R}^n)\,.$$

(iii) *If* $u_0\,p_1 \neq u_1\,p_0$ *then*

$$\big[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n)\big]^{1-\Theta}\,\big[\mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)\big]^{\Theta} \neq \mathcal{M}^{u}_{p}(\mathbb{R}^n)\,.$$

3.5 The $\pm$-method of Gustavsson-Peetre

- Gustavsson, Peetre (1977)
- Berezhnoi, Nilsson, Ovchinnikov, Shestakov

(X_0, X_1) - quasi-Banach couple, $\Theta \in (0,1)$.
$a \in \langle X_0, X_1, \Theta \rangle$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ with convergence in $X_0 + X_1$ and for **any finite subset** $N \subset \mathbb{Z}$ and any bounded sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$,

$$\Big\| \sum_{i\in N} \varepsilon_i \, 2^{i(j-\Theta)} \, a_i \Big\|_{X_j} \le C \sup_{i\in\mathbb{Z}} |\varepsilon_i|$$

for some constant C independent of N and $j \in \{0,1\}$.

$$\|a\|_{\langle X_0, X_1, \Theta\rangle} := \inf C \, .$$

Nilsson (1985):
Let X_0 and X_1 be two quasi-Banach lattices of type $\mathfrak{E}$ and $\Theta \in (0,1)$. Then

$$X_0^{1-\Theta}\, X_1^{\Theta} \hookrightarrow \langle X_0, X_1, \Theta\rangle \hookrightarrow (X_0^{1-\Theta}\, X_1^{\Theta})^{\sim}.$$

• Morrey spaces have the property $\mathfrak{E}$.
• $X^{\sim}$ - Gagliardo closure of X given by the collection of all $a \in X_0 + X_1$ such that there exists a sequence $\{a_i\}_{i\in\mathbb{N}_0} \subset X$ satisfying $a_i \to a$ as $i \to \infty$ in $X_0 + X_1$ and $\|a_i\|_X \le \lambda$ for some $\lambda < \infty$ and all $i \in \mathbb{N}_0$. We put $\|a\|_{X^{\sim}} := \inf \lambda$.

Theorem 2

Let $\Theta \in (0,1)$, $0 < p_0 \le u_0 < \infty$ *and* $0 < p_1 \le u_1 < \infty$. *Let* $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ *and* $p_0\, u_1 = p_1\, u_0$.
Then

$$\langle \mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n), \Theta \rangle = \mathcal{M}^{u}_{p}(\mathbb{R}^n).$$

- $\langle X_0, X_1, \Theta \rangle$ has the interpolation property.
- Lu, Yang, Yuan (2014), Yuan, S., Yang (2015).

Lemarié-Rieusset (2014):

$$[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]^{\Theta} = \mathcal{M}^{u}_{p}(\mathbb{R}^n)$$

Why $p_0\, u_1 = p_1\, u_0$?

Positive results are connected with the restriction

$$p_0\, u_1 = p_1\, u_0 \, .$$

Lemarié-Rieussiet (2014):

The mapping

$$T_\delta : \;\; f \mapsto (\arg f)\, |f|^\delta$$

is a bijection from $\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n)$ onto $\mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)$ if $\delta = p_0/p_1 = u_0/u_1$.

4. Some subspaces of $\mathcal{M}^u_p(\mathbb{R}^n)$

Let X be a quasi-Banach space of distributions or functions.

(i) By $\overset{\diamond}{X}$ we denote the closure in X of the set of all infinitely differentiable functions f such that $D^\alpha f \in X$ for all $\alpha \in \mathbb{N}_0^n$.

(ii) Let $C_0^\infty(\mathbb{R}^n) \hookrightarrow X$. Then we denote by $\mathring{X}$ the closure of $C_0^\infty(\mathbb{R}^n)$ in X.

Remark

There are explicit descriptions of $\mathring{\mathcal{M}}^u_p(\mathbb{R}^n)$ and $\overset{\diamond}{\mathcal{M}}{}^u_p(\mathbb{R}^n)$ in terms of the behaviour on balls.

Lemma 1

Let $1 \le p_0 < u_0 < \infty$, $1 \le p_1 \le u_1 < \infty$, $\Theta \in (0,1)$,

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \textit{and} \qquad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}\,.$$

Assume $p_0 < p_1$ *and* $p_0\, u_1 = p_1\, u_0$. *Then*

$$\mathring{\mathcal{M}}^u_p(\mathbb{R}^n) \hookrightarrow [\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\Theta \hookrightarrow \overset{\diamond}{\mathcal{M}}{}^u_p(\mathbb{R}^n)$$

and these embeddings are proper.

- We need new spaces for the description of

$$[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\Theta\ !$$

$\mathcal{M}^{u_0,u_1,\Theta}_{p_0,p_1}(\mathbb{R}^n)$ is the collection of all $f \in L^{\ell oc}_{p_1}(\mathbb{R}^n)$ such that

$$I_1(f) := \sup_{y\in\mathbb{R}^n} \sup_{0<r<1} |B(y,r)|^{1/u-1/p} \left[\int_{B(y,r)} |f(x)|^p \, dx\right]^{1/p} < \infty \, ,$$

$$\lim_{r\downarrow 0} |B(y,r)|^{1/u-1/p} \left[\int_{B(y,r)} |f(x)|^p \, dx\right]^{1/p} = 0$$

uniformly in $y \in \mathbb{R}^n$,

$$I_2(f) := \sup_{y\in\mathbb{R}^n} \sup_{r\geq 1} |B(y,r)|^{\frac{1}{u_0}-\frac{1}{p_0}} \left[\int_{B(y,r)} |f(x)|^{p_0} \, dx\right]^{1/p_0} < \infty$$

and

$$I_3(f) := \sup_{y\in\mathbb{R}^n} \sup_{r\geq 1} |B(y,r)|^{\frac{1}{u_1}-\frac{1}{p_1}} \left[\int_{B(y,r)} |f(x)|^{p_1} \, dx\right]^{1/p_1} < \infty \, .$$

Define

$$\| f \|_{\mathcal{M}^{u_0,u_1,\Theta}_{p_0,p_1}(\mathbb{R}^n)} := I_1(f) + I_2(f) + I_3(f)\,.$$

Theorem 3

Let $\Theta \in (0,1)$, $0 < p_i < u_i < \infty$, $i \in \{0,1\}$, $1 \le p_0 < p_1$ and $p_0\, u_1 = p_1\, u_0$. Define

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \text{and} \qquad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}\,.$$

Then

$$[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\Theta = \mathcal{M}^{u_0,u_1,\Theta}_{p_0,p_1}(\mathbb{R}^n)\,.$$

5. Spaces on $(0,1)^n$

Good news from the local case

Theorem 4

Let $\Theta \in (0,1)$, $1 \le p_0 \le u_0 < \infty$ *and* $1 \le p_1 \le u_1 < \infty$. *If* $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ *and* $p_0\, u_1 = p_1\, u_0$, *then*

$$[\mathcal{M}^{u_0}_{p_0}((0,1)^n), \mathcal{M}^{u_1}_{p_1}((0,1)^n)]_\Theta = \overset{\diamond}{\mathcal{M}}{}^u_p((0,1)^n).$$

6. Smoothness spaces built on $\mathcal{M}_p^u(\mathbb{R}^n)$

Sobolev-Morrey spaces: $1 \le p \le \infty$, $m \in \mathbb{N}$

$$f \in W^m(\mathcal{M}_p^u(\mathbb{R}^n)) \qquad \Longleftrightarrow \qquad D^\alpha f \in \mathcal{M}_p^u(\mathbb{R}^n)\,, \quad |\alpha| \le m\,.$$

$1 \le p \le \infty$, $0 < q \le \infty$:

$$(W_p^m(\mathbb{R}^n), L_p(\mathbb{R}^n))_{\Theta,q} = B_{p,q}^s(\mathbb{R}^n)\,, \qquad s := (1-\Theta)m\,.$$

$1 \le p \le u < \infty$, $0 < q \le \infty$:

$$(W^m(\mathcal{M}_p^u(\mathbb{R}^n)), \mathcal{M}_p^u(\mathbb{R}^n))_{\Theta,q} = \mathcal{N}_{u,p,q}^s(\mathbb{R}^n)\,, \qquad s := (1-\Theta)m\,.$$

Kozono/Yamazaki (1994).

The space $\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)} := \left\{ \sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|^q_{\mathcal{M}^u_p(\mathbb{R}^n)} \right\}^{1/q} < \infty.$$

- $\mathcal{N}^s_{p,p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$.
- $\mathcal{N}^{s_0}_{u,p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^{s_1}_{u,p,q}(\mathbb{R}^n), \qquad s_1 < s_0.$
- $\mathcal{N}^{s}_{u,p,q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^{s}_{u,p,q_1}(\mathbb{R}^n), \qquad q_0 < q_1.$

Theorem 5

Let $\Theta \in (0,1)$, $s_i \in \mathbb{R}$, $p_i,\ q_i \in (0,\infty]$ *and* $u_i \in [p_i,\infty]$, $i \in \{0,1\}$, *such that* $s = (1-\Theta)s_0 + \Theta s_1$, (1), (2) *and* $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$.
If $p_0\, u_1 = p_1\, u_0$, *then*

$$\left\langle \mathcal{N}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^n), \mathcal{N}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n), \Theta \right\rangle = \mathcal{N}^{s}_{u,p,q}(\mathbb{R}^n)\,,$$

$$\begin{aligned}
\mathring{\mathcal{N}}^{s}_{u,p,q}(\mathbb{R}^n) &= [\mathring{\mathcal{N}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^n), \mathring{\mathcal{N}}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n)]_\Theta \\
&= [\mathcal{N}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^n), \mathring{\mathcal{N}}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n)]_\Theta \\
&= [\mathring{\mathcal{N}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^n), \mathcal{N}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^n)]_\Theta
\end{aligned}$$

and, if $\min(p_0, p_1, q_0, q_1) \geq 1$,

$$[\mathcal{N}^{s_0}_{u_0,p_0,q_0}((0,1)^n), \mathcal{N}^{s_1}_{u_1,p_1,q_1}((0,1)^n)]_\Theta = \overset{\diamond}{\mathcal{N}}{}^{s}_{u,p,q}((0,1)^n)$$

Special case: $u_0 = p_0,\ u_1 = p_1$

$$\langle B^{s_0}_{p_0,q_0}(\mathbb{R}^n), B^{s_1}_{p_1,q_1}(\mathbb{R}^n), \Theta\rangle = B^s_{p,q}(\mathbb{R}^n)$$

$$\begin{aligned} \mathring{B}^s_{p,q}(\mathbb{R}^n) &= [\mathring{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n), \mathring{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n)]_\Theta \\ &= [\mathring{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n), B^{s_1}_{p_1,q_1}(\mathbb{R}^n)]_\Theta . \end{aligned}$$

Theorem 6

It holds

$$B^s_{p,q}(\mathbb{R}^n) = [B^{s_0}_{p_0,q_0}(\mathbb{R}^n), B^{s_1}_{p_1,q_1}(\mathbb{R}^n)]_\Theta$$

if and only if either $B^{s_0}_{p_0,q_0}(\mathbb{R}^n) = B^{s_1}_{p_1,q_1}(\mathbb{R}^n)$ *or at least one of the spaces* $B^{s_0}_{p_0,q_0}(\mathbb{R}^n)$, $B^{s_1}_{p_1,q_1}(\mathbb{R}^n)$ *is separable, i.e.,*

$$\min\Big(\max(p_0, q_0), \max(p_1, q_1)\Big) < \infty\,.$$

Theorem 6 yields:

$$B^s_{p,\infty}(\mathbb{R}^n) \neq [B^{s_0}_{p_0,\infty}(\mathbb{R}^n), B^{s_1}_{p_1,\infty}(\mathbb{R}^n)]_\Theta$$

$$B^s_{\infty,q}(\mathbb{R}^n) \neq [B^{s_0}_{\infty,q_0}(\mathbb{R}^n), B^{s_1}_{\infty,q_1}(\mathbb{R}^n)]_\Theta$$

- Triebel (1978): $1 < p_0 = p_1 < \infty$, $s_0 \neq s_1$

$$\mathring{B}^s_{p,\infty}(\mathbb{R}^n) = [B^{s_0}_{p,\infty}(\mathbb{R}^n), B^{s_1}_{p,\infty}(\mathbb{R}^n)]_\Theta$$

Theorem 7

(i) *Let* $0 < p_0 < p_1 \leq \infty$ *and* $q_0 = q_1 = \infty$. *If* $s_0 - n/p_0 > s_1 - n/p_1$, *then*

$$[B^{s_0}_{p_0,\infty}(\mathbb{R}^n), B^{s_1}_{p_1,\infty}(\mathbb{R}^n)]_\Theta = \mathring{B}^{s}_{p,\infty}(\mathbb{R}^n)\,.$$

(ii) *If either* $s_0 \neq s_1$ *or* $s_0 = s_1$ *and* $q_0 \neq q_1$, *then*

$$[B^{s_0}_{\infty,q_0}(\mathbb{R}^n), B^{s_1}_{\infty,q_1}(\mathbb{R}^n)]_\Theta = \overset{\diamond}{B}{}^{s}_{\infty,q}(\mathbb{R}^n)\,.$$

- Part (i): S., Skrzypczak, Vybiral (2014).
- Part (ii): Yuan, S., Yang (2015).

Thank you very much for your attention !