On interpolation of spaces of integrable functions with respect to a vector measure

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• Let (Ω, Σ) be a measurable space and μ a σ-finite measure on (Ω, Σ). If $1 \leq p_0 \neq p_1 \leq \infty$, $0 < \theta < 1$, $0 < q \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$(L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,q} = L^{p,q}(\mu),$$

with equivalence of quasi-norms. In particular,

$$(L^1(\mu), L^\infty(\mu))_{1-\frac{1}{p},p} = L^p(\mu), \quad 1 < p < \infty.$$

• If m is a vector measure, then a similar result does not hold. Thus,

$$(L^1(m), L^\infty(m))_{1-\frac{1}{p},p} \subsetneq L^p(m), \quad 1 < p < \infty.$$

The inclusion $L^\infty(m) \subseteq L^1(m)$ is weakly compact and thus, by Beauzamy's result, $(L^1(m), L^\infty(m))_{1-\frac{1}{p},p}$ is reflexive for $1 < p < \infty$.

THEOREM (Beauzamy, Lecture Notes in Math. (1978))

Let $0 < \theta < 1$ and $1 < q < \infty$.

$$(A_0, A_1)_{\theta,q} \text{ is reflexive } \Leftrightarrow I : A_0 \cap A_1 \longrightarrow A_0 + A_1 \text{ is weakly compact.}$$

However, $L^p(m)$, $p > 1$, is not reflexive whenever $L^1(m) \neq L^1_w(m)$.

A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.

Theorem (Fernández, Mayoral and Naranjo, J. Math. Anal. Appl. (2011))

If $0<\theta<1$, $0<q\le\infty$ and $\frac{1}{p}=1-\theta$, it holds

$$\big(L^1(m),L^\infty(m)\big)_{\theta,q}=\big(L^1_w(m),L^\infty(m)\big)_{\theta,q}=L^{p,q}(\|m\|).$$

- The **Lorentz space** $\Lambda^q_\varphi(\|m\|)$

For $0<q\le\infty$ and a non-negative function φ on $(0,\infty)$, $\Lambda^q_\varphi(\|m\|)$ is the space of (m-a.e. equivalence classes of) scalar measurable functions on Ω s.t.

$$\|f\|_{\Lambda^q_\varphi(\|m\|)}:=\left(\int_0^\infty(\varphi(t)f_*(t))^q\,\frac{dt}{t}\right)^{\frac{1}{q}}<\infty$$

(with the usual modification for $q=\infty$). Here f_* is the decreasing rearrangement (w.r.t. m) of f given by

$$f_*(t):=\inf\{s>0:\|m\|(\{w\in\Omega:|f(w)|>s\})\le t\},$$

and $\|m\|(A):=\sup\{|\langle m,x^*\rangle|(A):x^*\in B(X^*)\}$ the semivariation of m.

If $\varphi(t)=t^{1/p}$, $\Lambda^q_\varphi(\|m\|)=L^{p,q}(\|m\|)$.

• Let Ω be non-empty set, Σ a σ-algebra of Ω and X a Banach space. Let $m : \Sigma \to X$ be a countably additive **vector measure**.

$L^0(m)$ denotes the space of all scalar measurable functions on Ω.
$f, g \in L^0(m)$ will be identified if are equal m-a.e., that is, whenever

$$\|m\|(\{w \in \Omega : f(w) \neq g(w)\}) = 0.$$

• $f \in L^0(m)$ is called **integrable** (w.r.t. m) if

i) $f \in L^1(|\langle m, x^* \rangle|)$, for all $x^* \in X^*$ (i.e. f is **weakly integrable** w.r.t. m)

ii) given any $A \in \Sigma$, there exists an element $\int_A fdm \in X$ such that $\left\langle \int_A fdm, x^* \right\rangle = \int_A f\, d\langle m, x^* \rangle$, for all $x^* \in X^*$.

Let

$L^1_w(m) := \{f : f \text{ is weakly integrable}\}$,

$L^1(m) := \{f : f \text{ is integrable}\}$,

endowed with the norm

$$\|f\|_1 := \sup \left\{ \int_\Omega |f|\, d\, |\langle m, x^* \rangle| : x^* \in B(X^*) \right\}.$$

- Given $1 < p < \infty$, $f \in L^0(m)$ is said to be
  - i) **weakly p-integrable** (w.r.t. m) if $|f|^p \in L^1_w(m)$,
  - ii) **p-integrable** (w.r.t. m) if $|f|^p \in L^1(m)$,

Let
$$L^p_w(m) := \{f : f \text{ is weakly } p\text{-integrable}\},$$
$$L^p(m) := \{f : f \text{ is } p\text{-integrable}\},$$
with the norm
$$\|f\|_p := \sup\left\{\left(\int_\Omega |f|^p \, d\,|\langle m, x^*\rangle|\right)^{1/p} : x^* \in B(X^*)\right\}.$$

- Some properties:
  - $L^p(m)$ is a Banach lattice with order continuous norm.
  - $L^p_w(m)$ is a Banach lattice with the Fatou property.
  - $L^p(m)$ and $L^p_w(m)$ may not be reflexive for $p > 1$.
  - If $1 < p_1 < p_2 < \infty$, then $L^\infty(m) \subseteq L^{p_2}(m) \subseteq L^{p_2}_w(m) \subseteq L^{p_1}(m) \subseteq L^{p_1}_w(m) \subseteq L^1(m) \subseteq L^1_w(m)$.

A. Fernández, F. Mayoral, F. Naranjo, C. Sáez and E.A. Sánchez–Pérez, *Spaces of p-integrable functions with respect to a vector measure*, Positivity **10** (2006), 1–16.

• When m is a finite positive scalar measure, $\|m\|$ and m coincide. But in general, for an arbitrary vector measure m, it holds that

$$L^p(m) \neq L^p(\|m\|) := L^{p,p}(\|m\|), \ 1 \leq p < \infty.$$

We have the following continuous inclusions:

$$\begin{aligned} L^\infty(m) \subseteq L^{p,1}(\|m\|) &\subseteq L^p(\|m\|) \subseteq L^p(m) \\ &\subseteq L^p(m) \subseteq L^p_w(m) \subseteq L^{p,\infty}(\|m\|) \subseteq L^{1,\infty}(\|m\|). \end{aligned}$$

- A non-negative function ρ defined on $\mathbb{R}^+ := (0, \infty)$ belongs to the class $Q(0,1)$ if there exists $0 < \varepsilon < 1/2$ such that

$$\rho(t)t^{-\varepsilon} \text{ is non-decreasing } (\uparrow) \quad \text{and} \quad \rho(t)t^{-(1-\varepsilon)} \text{ is non-increasing } (\downarrow).$$

L.E. Persson, *Interpolation with a parameter function*, Math. Scand. **59** (1986), 199–222.

- For a quasi-Banach couple (X_0, X_1), the **real interpolation space** $(X_0, X_1)_{\rho,q}$, $\rho \in Q(0,1)$, $0 < q \leq \infty$, consists of all $x \in X_0 + X_1$ for which

$$\|x\|_{\rho,q} := \left(\int_0^\infty \left[\frac{K(t, x; X_0, X_1)}{\rho(t)} \right]^q \frac{dt}{t} \right)^{1/q} < \infty,$$

(with the usual modification for $q = \infty$), where the **K-functional** is defined for $t > 0$ as

$$K(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} \; : \; x = x_0 + x_1, \; x_i \in X_i\}, \, x \in X_0 + X_1.$$

- When $\rho(t) = t^\theta$, $0 < \theta < 1$, we get the classical space $(X_0, X_1)_{\theta,q}$.
- It holds that

$$(L^1(\mu), L^\infty(\mu))_{\rho(t)=t^{1-\frac{1}{p}}(1+|\log t|)^{-\alpha},q} = L^{p,q}(\log L)^\alpha(\mu).$$

Other similar classes of functions

- B_K : $\rho \in C(\mathbb{R}^+)$ non-decreasing such that

$$\bar{\rho}(t) = \sup_{s>0} \frac{\rho(ts)}{\rho(s)} < \infty \quad \text{for every } t > 0,$$

$$\int_0^\infty \min\left\{1, \frac{1}{t}\right\} \bar{\rho}(t) \frac{dt}{t} < \infty.$$

- B_ψ : $\rho \in C^1(\mathbb{R}^+)$ satisfying

$$0 < \inf_{t>0} \frac{t\rho'(t)}{\rho(t)} \leq \sup_{t>0} \frac{t\rho'(t)}{\rho(t)} < 1.$$

- $\mathcal{P}^{+-}$: $\rho(t)$ non-decreasing, $\rho(t)/t$ non-increasing and

$$\bar{\rho}(t) = o(\max\{1, t\}) \quad \text{as } t \to 0 \text{ and } t \to \infty.$$

Proposition (Gustavsson, Math. Scand. (1978) / Persson, Math. Scand. (1986))

a) $B_\psi \subseteq Q(0,1) \subseteq \mathcal{P}^{+-}$.

b) $B_\psi \subseteq B_K \subseteq \mathcal{P}^{+-}$.

c) If $\rho \in \mathcal{P}^{+-}$, there exists $\varphi \in B_\psi$ such that $\rho \approx \varphi$.

R. Campo, A. Fernández, A. M., F. Mayoral and F. Naranjo, *Interpolation with a parameter function and integrable function spaces with respect to vector measures*, Math. Ineq. Appl. **18** (2015), 707–720.

- The K-functional for $(L^1(\|m\|), L^\infty(m))$ and $(L^{1,\infty}(\|m\|), L^\infty(m))$:

Proposition

For each $f \in L^1(\|m\|)$,

$$K\left(t, f; L^1(\|m\|), L^\infty(m)\right) = \int_0^\infty \min\{t, \|m\|_f(s)\}\, ds = \int_0^t f_*(s) ds,$$

where $\|m\|_f(t) := \|m\|(\{w \in \Omega : |f(w)| > t\})$.

Proposition

It holds that

$$\sup_{s>0} s \min\{t, \|m\|_f(s)\} \preceq K\left(t, f; L^{1,\infty}(\|m\|), L^\infty(m)\right),$$

for all $f \in L^{1,\infty}(\|m\|)$. In particular (taking $s := f_*(t)/2$),

$$t f_*(t) \preceq K\left(t, f; L^{1,\infty}(\|m\|), L^\infty(m)\right).$$

The next result follows from these estimates for the K-functional and weighted Hardy's inequality for non-increasing functions:

Theorem

Let $\rho \in Q(0,1)$, $0 < q \leq \infty$ and $\varphi(t) = \dfrac{t}{\rho(t)}$. Then,

$$\big(L^1(\|m\|), L^\infty(m)\big)_{\rho,q} = \big(L^{1,\infty}(\|m\|), L^\infty(m)\big)_{\rho,q} = \Lambda_\varphi^q(\|m\|).$$

In particular, if $0 < \theta < 1$, it holds that

$$\big(L^1(\|m\|), L^\infty(m)\big)_{\theta,q} = \big(L^{1,\infty}(\|m\|), L^\infty(m)\big)_{\theta,q} = L^{\frac{1}{1-\theta},q}(\|m\|).$$

Using the last theorem, reiteration and the continuous inclusions

$$L^\infty(m) \subseteq L^r(\|m\|) \subseteq L^r(m) \subseteq L^r_w(m) \subseteq L^{r,\infty}(\|m\|), \quad r \geq 1,$$

Theorem

If $1 \leq p_0 \neq p_1 \leq \infty$, $\rho \in Q(0,1)$, $\varphi(t) = \dfrac{t^{\frac{1}{p_0}}}{\rho\left(t^{\frac{1}{p_0}-\frac{1}{p_1}}\right)}$ and $0 < q \leq \infty$,

$$(L^{p_0}(m), L^{p_1}(m))_{\rho,q} = (L^{p_0}_w(m), L^{p_1}(m))_{\rho,q} = (L^{p_0}_w(m), L^{p_1}_w(m))_{\rho,q} = \Lambda^q_\varphi(\|m\|).$$

In particular, if $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, it holds that

$$(L^{p_0}(m), L^{p_1}(m))_{\theta,q} = (L^{p_0}_w(m), L^{p_1}(m))_{\theta,q} = (L^{p_0}_w(m), L^{p_1}_w(m))_{\theta,q} = L^{p,q}(\|m\|).$$

Corollary

For $\rho(t) = t^{1-\frac{1}{p}}(1+|\log t|)^{-\alpha}$, $1 < p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$,

$$\left(L^1(m), L^\infty(m)\right)_{\rho,\,q} = \left(L^1_w(m), L^\infty(m)\right)_{\rho,q} = L^{p,q}(\log L)^\alpha(\|m\|).$$

When $\varphi(t) = t^{\frac{1}{p}}(1+|\log t|)^\alpha$, $\Lambda^q_\varphi(\|m\|) = L^{p,q}(\log L)^\alpha(\|m\|)$, that can be considered the version of Lorentz-Zygmund space in the vector case.

Theorem

Let $\rho \in Q(0,1)$, $1 \leq p < \infty$ and $0 < q_0, q, q_1 \leq \infty$.

a) If $\varphi_0 \in Q(0,1)$,

$$(\Lambda^{q_0}_{\varphi_0}(\|m\|), L^\infty(m))_{\rho,q} = \Lambda^q_\varphi(\|m\|), \quad \varphi(t) = \frac{\varphi_0(t)}{\rho(\varphi_0(t))}.$$

b) If $\varphi_1 \in Q(0,1/p)$ (i.e. $\varphi_1(t)t^{-\varepsilon} \uparrow$ and $\varphi_1(t)t^{-\left(\frac{1}{p}-\varepsilon\right)} \downarrow$ for some $0 < \varepsilon < \frac{1}{2p}$),

$$(L^p(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|))_{\rho,q} = \Lambda^q_\varphi(\|m\|), \quad \varphi(t) = \frac{t^{1/p}}{\rho(t^{1/p}/\varphi_1(t))}.$$

c) If $\varphi_i \in Q(0,1)$, $i = 0,1$, and $\phi := \frac{\varphi_0}{\varphi_1} \in Q(0,b)$ for some $b \in \mathbb{R}$ (i.e. $\phi(t)t^{-\varepsilon} \uparrow$ and $\phi(t)t^{-(b-\varepsilon)} \downarrow$ for some $0 < \varepsilon < \frac{b}{2}$),

$$(\Lambda^{q_0}_{\varphi_0}(\|m\|), \Lambda^{q_1}_{\varphi_1}(\|m\|))_{\rho,q} = \Lambda^q_\varphi(\|m\|), \quad \varphi(t) = \frac{\varphi_0(t)}{\rho(\varphi_0(t)/\varphi_1(t))}.$$

A. Fernández, F. Mayoral, F. Naranjo and E. A. Sánchez–Pérez, *Complex interpolation of spaces of integrable functions with respect to a vector measure*, Collect. Math. **61** (2010), 241–252.

Theorem (Fernández, Mayoral, Naranjo and Sánchez-Pérez, Collect. Math. (2010))

Given $1 \leq p_0 \neq p_1 \leq \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, it holds that

$$[L^{p_0}(m), L^{p_1}(m)]_{[\theta]} = [L^{p_0}_w(m), L^{p_1}(m)]_{[\theta]} = [L^{p_0}_w(m), L^{p_1}_w(m)]_{[\theta]} = L^p(m),$$

$$[L^{p_0}(m), L^{p_1}(m)]^{[\theta]} = [L^{p_0}_w(m), L^{p_1}(m)]^{[\theta]} = [L^{p_0}_w(m), L^{p_1}_w(m)]^{[\theta]} = L^p_w(m).$$

R. Campo, A. Fernández, A. M., F. Mayoral and F. Naranjo, *Complex interpolation of Orlicz spaces with respect to a vector measure*, Math. Nachr. **287** (2014), 23–31.

• Orlicz spaces $L^{\phi}(m)$ and $L^{\phi}_{w}(m)$ generalize the spaces $L^{p}(m)$ and $L^{p}_{w}(m)$, respectively. We are interested in studying if the following equalities hold:

$$[L^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = [L^{\phi_0}_{w}(m), L^{\phi_1}(m)]_{[\theta]} = [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]_{[\theta]} = L^{\phi}(m),$$

$$[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L^{\phi_0}_{w}(m), L^{\phi_1}(m)]^{[\theta]} = [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]^{[\theta]} = L^{\phi}_{w}(m).$$

R. Campo, A. Fernández, A. M., F. Mayoral and F. Naranjo, *Complex interpolation of Orlicz spaces with respect to a vector measure*, Math. Nachr. **287** (2014), 23–31.

• Given $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$, $\phi^{-1} = (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$, do the following equalities hold?

$$[L^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = [L_w^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]_{[\theta]} = L^{\phi}(m),$$

$$[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L_w^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m).$$

O. Delgado, *Banach function subspaces of L^1 of a vector measure and related Orlicz spaces*, Indag. Math. **15** (2004), 485–495.

- An N-function is any function $\phi : [0,\infty) \to [0,\infty)$ which is

  - strictly increasing,
  - $\phi(0) = 0$,
  - continuous,
  - $\displaystyle \lim_{x\to 0} \frac{\phi(x)}{x} = 0$,
  - convex,
  - $\displaystyle \lim_{x\to \infty} \frac{\phi(x)}{x} = \infty$.

An N-function has the Δ_2-property (we write $\phi \in \Delta_2$) if

$$\exists C > 0 \text{ such that } \phi(2x) \leq C\phi(x) \text{ for all } x \geq 0.$$

- The **weak Orlicz space** $L_w^\phi(m)$ (w.r.t. m and ϕ) is defined as

$$L_w^\phi(m) := \left\{ f \in L^0(m) : \|f\|_{L_w^\phi(m)} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{L_w^\phi(m)} &:= \sup\left\{ \|f\|_{L^\phi(|\langle m, x^*\rangle|)} : x^* \in B_{X^*} \right\} \\ &= \sup_{x^* \in B_{X^*}} \inf\left\{ k > 0 : \int_\Omega \phi\left(\frac{|f|}{k}\right) d|\langle m, x^*\rangle| \le 1 \right\}. \end{aligned}$$

$L_w^\phi(m)$ coincides with the intersection of all Orlicz $L^\phi(|\langle m, x^*\rangle|)$, $x^* \in X^*$.

- The **Orlicz space** $L^\phi(m)$ (w.r.t. m and ϕ) is defined by $\overline{\mathcal{S}(\Sigma)}^{L_w^\phi(m)}$.
- If $\phi(x) = x^p$, $L_w^\phi(m)$ and $L^\phi(m)$ correspond to $L_w^p(m)$ and $L^p(m)$, respect.

- The corresponding **Orlicz classes** (w.r.t. m and ϕ) are given by

$$O_w^\phi(m) := \{f :\in L^0(m) : \phi(|f|) \in L_w^1(m)\},$$

$$O^\phi(m) := \{f :\in L^0(m) : \phi(|f|) \in L^1(m)\}.$$

It holds that

$$O_w^\phi(m) \subseteq L_w^\phi(m) \text{ and } O^\phi(m) \subseteq L^\phi(m).$$

- The **weak Orlicz space** $L_w^\phi(m)$ (w.r.t. m and ϕ) is defined as

$$L_w^\phi(m) := \left\{ f \in L^0(m) : \|f\|_{L_w^\phi(m)} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{L_w^\phi(m)} &:= \sup\left\{ \|f\|_{L^\phi(|\langle m, x^*\rangle|)} : x^* \in B_{X^*} \right\} \\ &= \sup_{x^* \in B_{X^*}} \inf\left\{ k > 0 : \int_\Omega \phi\left(\frac{|f|}{k}\right) d|\langle m, x^*\rangle| \leq 1 \right\}. \end{aligned}$$

$L_w^\phi(m)$ coincides with the intersection of all Orlicz $L^\phi(|\langle m, x^*\rangle|)$, $x^* \in X^*$.

- The **Orlicz space** $L^\phi(m)$ (w.r.t. m and ϕ) is defined by $\overline{\mathcal{S}(\Sigma)}^{L_w^\phi(m)}$.
- If $\phi(x) = x^p$, $L_w^\phi(m)$ and $L^\phi(m)$ correspond to $L_w^p(m)$ and $L^p(m)$, respect.

- The corresponding **Orlicz classes** (w.r.t. m and ϕ) are given by

$$O_w^\phi(m) := \{f :\in L^0(m) : \phi(|f|) \in L_w^1(m)\},$$

$$O^\phi(m) := \{f :\in L^0(m) : \phi(|f|) \in L^1(m)\}.$$

When $\phi \in \Delta_2$

$$O_w^\phi(m) = L_w^\phi(m) \text{ and } O^\phi(m) = L^\phi(m).$$

• Let (X_0, X_1) be a couple of Banach lattices on the same measure space and $0 < \theta < 1$, the **Calderón's space** $X_0^{1-\theta} X_1^{\theta}$ is

$$X_0^{1-\theta} X_1^{\theta} := \{f \in L^0 : \exists \lambda > 0, \exists f_i \in B_{X_i} \text{ s.t. } |f| \leq \lambda |f_0|^{1-\theta} |f_1|^{\theta}\},$$

with the norm

$$\|f\|_{X_0^{1-\theta} X_1^{\theta}} := \inf\{\lambda > 0 : |f| \leq \lambda |f_0|^{1-\theta} |f_1|^{\theta},\ f_0 \in B_{X_0},\ f_1 \in B_{X_1}\}.$$

It holds that

C1 $X_0 \cap X_1 \subseteq [X_0, X_1]_{[\theta]} \subseteq X_0^{1-\theta} X_1^{\theta} \subseteq [X_0, X_1]^{[\theta]} \subseteq X_0 + X_1.$

C2 If X_0 or X_1 is order continuous, then $[X_0, X_1]_{[\theta]} = X_0^{1-\theta} X_1^{\theta}$.

C3 If X_0 and X_1 have the Fatou property then $[X_0, X_1]^{[\theta]} = X_0^{1-\theta} X_1^{\theta}$.

• Given a Banach couple (X_0, X_1) and $0 < \theta < 1$, the **Gustavsson-Peetre space** $\langle X_0, X_1, \theta\rangle$ is the Banach space formed by

$$x \in X_0 + X_1 \text{ for which } \exists (x_k)_{k\in\mathbb{Z}} \subseteq X_0 \cap X_1 \text{ s.t.}$$

a) $x = \sum_{k\in\mathbb{Z}} x_k$, where the series converges in $X_0 + X_1$.

b) $\exists C > 0$ s.t. for every finite subset $F \subseteq \mathbb{Z}$ and every subset of scalars $(\varepsilon_k)_{k\in F}$, with $|\varepsilon_k| \leq 1$,

$$\left\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k\theta}} x_k \right\|_{X_0} \leq C \quad \text{and} \quad \left\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k(\theta-1)}} x_k \right\|_{X_1} \leq C.$$

The norm considered in $\langle X_0, X_1, \theta\rangle$ is

$$\|x\|_{\langle X_0, X_1, \theta\rangle} = \inf\{C > 0 : \text{taken over all } (x_k)_{k\in\mathbb{Z}} \text{ satisfying a) and b)}\}.$$

Moreover,

GP $\langle X_0, X_1, \theta\rangle \subseteq [X_0, X_1]^{[\theta]}$.

Proposition

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$. Then

(1) $L^{\phi_0}(m)^{1-\theta} L^{\phi_1}(m)^{\theta} = L^{\phi}(m)$.

(2) $L_w^{\phi_0}(m)^{1-\theta} L_w^{\phi_1}(m)^{\theta} = L_w^{\phi}(m)$.

$L^{\phi}(m)$ is order continuous and $L_w^{\phi}(m)$ has the Fatou property.

Corollary

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and ϕ s.t. $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$. It holds that

$$[L^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = L^{\phi}(m).$$

$$[L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m).$$

M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., 1991.

- Some partial ordering relations between N-functions:

  $\phi_1 \prec \phi_0$ if $\exists \varepsilon > 0$, $\exists x_0 \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_0$.

  $\phi_1 \prec\!\!\prec \phi_0$ if $\forall \varepsilon > 0$, $\exists x_\varepsilon \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_\varepsilon$.

Lemma

Let $\phi_0, \phi_1 \in \Delta_2$.

(1) If $\phi_1 \prec \phi_0$, then $L_w^{\phi_0}(m) \subseteq L_w^{\phi_1}(m)$, and $L^{\phi_0}(m) \subseteq L^{\phi_1}(m)$.

(2) If $\phi_1 \prec\!\!\prec \phi_0$, then $L_w^{\phi_0}(m) \subseteq L^{\phi_1}(m)$.

For $\phi_1(x) := x^p$, $\phi_0(x) := x^q$, $1 < p < q$, it follows that $\phi_1 \prec\!\!\prec \phi_0$, and therefore the well-known inclusion $L_w^q(m) \subseteq L^p(m)$.

M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., 1991.

- Some partial ordering relations between N-functions:

$\phi_1 \prec \phi_0$ if $\exists \varepsilon > 0$, $\exists x_0 \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_0$.

$\phi_1 \prec\prec \phi_0$ if $\forall \varepsilon > 0$, $\exists x_\varepsilon \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_\varepsilon$.

Lemma

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$.

(1) If $\phi_1 \prec \phi_0$, then $L_w^{\phi_0}(m) \subseteq L_w^{\phi_1}(m)$, and $L^{\phi_0}(m) \subseteq L^{\phi_1}(m)$.

(2) If $\phi_1 \prec\prec \phi_0$, then $L_w^{\phi_0}(m) \subseteq L^{\phi_1}(m)$.

(3) If $\phi_1 \prec \phi_0$, then $\phi_1 \prec \phi \prec \phi_0$. If $\phi_1 \prec\prec \phi_0$, then $\phi_1 \prec\prec \phi \prec\prec \phi_0$.

Theorem

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$.
If $\phi_1 \prec\prec \phi_0$, it follows that

$$\langle L^{\phi_0}(m), L^{\phi_1}(m), \theta \rangle = L_w^{\phi}(m).$$

$$\begin{aligned} L^{\phi}_w(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\ &\subseteq [L^{\phi_0}_w(m), L^{\phi_1}_w(m)]^{[\theta]} = (L^{\phi_0}_w(m))^{1-\theta}(L^{\phi_1}_w(m))^{\theta} = L^{\phi}_w(m). \end{aligned}$$

Therefore,

$$[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L^{\phi_0}_w(m), L^{\phi_1}_w(m)]^{[\theta]} = L^{\phi}_w(m),$$

and, by $L^{\phi_i}(m) \subseteq L^{\phi_i}_w(m)$ $(i = 0, 1)$, it also holds that

$$[L^{\phi_0}(m), L^{\phi_1}_w(m)]^{[\theta]} = [L^{\phi_0}_w(m), L^{\phi_1}(m)]^{[\theta]} = L^{\phi}_w(m).$$

This gives (i) in the following theorem.

Theorem

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$. If $\phi_1 \lll \phi_0$, then

(i) $[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L^{\phi_0}(m), L^{\phi_1}_w(m)]^{[\theta]} = [L^{\phi_0}_w(m), L^{\phi_1}(m)]^{[\theta]} = L^{\phi}_w(m)$.

(ii) $[L^{\phi_0}_w(m), L^{\phi_1}_w(m)]_{[\theta]} = [L^{\phi_0}(m), L^{\phi_1}_w(m)]_{[\theta]} = [L^{\phi_0}_w(m), L^{\phi_1}(m)]_{[\theta]} = L^{\phi}(m)$.

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