Embeddings of Morrey-type spaces

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joint work with S. Moura (Coimbra), L. Skrzypczak (Poznań),
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Starting point: Morrey Spaces

$0 < p \le u < \infty$, Morrey space $\mathcal{M}_{u,p}$: $f \in L_p^{\mathrm{loc}}$ with

$$\|f|\mathcal{M}_{u,p}\| = \sup_{x\in\mathbb{R}^n,\ R>0} R^{\frac{n}{u}-\frac{n}{p}} \left(\int_{B(x,R)} |f(y)|^p \, \mathrm{d}y \right)^{1/p} < \infty$$

Rem. Ch. Morrey (1938), Campanato, Brudnyi, Peetre (1960's), Burenkov, Guliev, Goldman, Tararykova, Nursultanov, variety of notation/concepts, ...

Some properties:

- $\mathcal{M}_{u,p} = \begin{cases} \{0\}, & p > u \\ L_p, & p = u \end{cases}$ $\curvearrowright$ $u > p$ refined (local) integrability
- $L_u = \mathcal{M}_{u,u} \hookrightarrow \mathcal{M}_{u,p_1} \hookrightarrow \mathcal{M}_{u,p_2}$, $0 < p_2 \le p_1 \le u < \infty$
- $\|f(\lambda\,\cdot)|\mathcal{M}_{u,p}\| = \lambda^{-n/u}\|f|\mathcal{M}_{u,p}\|$, $\lambda > 0$

All spaces are defined on $\mathbb{R}^n$.

Starting point: Morrey Spaces

A first example

Let $0 < p < u < \infty$.

- $f(x) = |x|^{-\frac{n}{u}} \begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

  $\curvearrowright$ $f \in \mathcal{M}_{u,p}$, but $f \notin L_u$ $\dashrightarrow$ $L_u \subsetneq \mathcal{M}_{u,p}$

- Let $m_k = (2^k, 0, \cdots, 0)$, $k \in \mathbb{N}_0$

$$\curvearrowright \; g(x) = \sum_{k=0}^{\infty} f(x - m_k) \; \in \; \mathcal{M}_{u,p} \setminus L_u$$

Rem. $L_{u,\infty} \subsetneq \mathcal{M}_{u,p}, \quad 0 < p < u < \infty$

Classical Smoothness Spaces

Spaces of Besov and Sobolev type

$0 < p \leq \infty,\ 0 < q \leq \infty,\ s \in \mathbb{R},\ \{\varphi_j\}_j$ dyadic partition of unity

$$\left\| f | B^s_{p,q} \right\| = \left\| \left(2^{js} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p \right\| \right)_j | \ell_q \right\|$$

$$\left\| f | F^s_{p,q} \right\| = \left\| \left\| \left(2^{js} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot) \right)_j | \ell_q \right\| \, | L_p \right\|$$

Rem.

- (classical) Besov spaces $0 < p, q \leq \infty,\ s > n\left(\frac{1}{p} - 1\right)_+$

$$\left\| f | B^s_{p,q} \right\| \sim \| f \mid L_p \| + \left(\int_0^1 \left(\frac{\omega_m(f,t)_p}{t^s} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q},\ m > s$$

- $B^s_{\infty,\infty} = \mathcal{C}^s,\ \ s > 0$ Hölder-Zygmund spaces
- $F^s_{p,2} = H^s_p,\ 1 < p < \infty,\ s \in \mathbb{R}$
  $F^k_{p,2} = W^k_p,\ k \in \mathbb{N}_0,\ 1 < p < \infty$ Sobolev spaces

Approach 1: Smoothness Morrey spaces

The spaces $\mathcal{N}^s_{u,p,q}$ and $\mathcal{E}^s_{u,p,q}$

$0 < p \leq u < \infty$, $q \in (0, \infty]$, $s \in \mathbb{R}$, $\{\varphi_j\}_j$ smooth dyadic res. of unity

(i) Besov-Morrey space $\mathcal{N}^s_{u,p,q}$: $f \in \mathcal{S}'$ with

$$\left\| f \mid \mathcal{N}^s_{u,p,q} \right\| = \left\| \left(2^{js} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | \mathcal{M}_{u,p} \right\| \right)_j | \ell_q \right\| < \infty$$

(ii) Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{u,p,q}$: $f \in \mathcal{S}'$ with

$$\left\| f \mid \mathcal{E}^s_{u,p,q} \right\| = \left\| \left\| \left(2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)| \right)_j | \ell_q \right\| \mid \mathcal{M}_{u,p} \right\| < \infty$$

Rem.

- Kozono/Yamazaki ('94), Mazzucato ('03), Tang/Xu ('05), Sawano ('07-)

H. Kozono and M. Yamazaki
Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data.
Comm. Partial Differential Equations, 19 (1994), 959–1014.

A.L. Mazzucato
Besov-Morrey spaces: function space theory and applications to non-linear PDE.
Trans. Amer. Math. Soc., 355 (2003), 1297–1364 (electronic).

Approach 1: Smoothness Morrey spaces

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(ii) Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{u,p,q}$: $f \in \mathcal{S}'$ with

$$\left\| f \mid \mathcal{E}^s_{u,p,q} \right\| = \left\| \left\| \left(2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)| \right)_j | \ell_q \right\| \mid \mathcal{M}_{u,p} \right\| < \infty$$

Rem.

- Kozono/Yamazaki ('94), Mazzucato ('03), Tang/Xu ('05), Sawano ('07-)
- $\mathcal{N}^s_{p,p,q} = B^s_{p,q}$, $\mathcal{E}^s_{p,p,q} = F^s_{p,q}$, $\mathcal{N}^s_{u,p,q} = \mathcal{E}^s_{u,p,q} = \{0\}$ if $p > u$
- elementary properties (quasi-Banach, monotonicity, $\mathcal{S} \hookrightarrow \cdots \hookrightarrow \mathcal{S}'$, ...)
- $\mathcal{E}^0_{u,p,2} = \mathcal{M}_{u,p}$, $1 < p \le u < \infty$, i.e., $\mathcal{E}^0_{p,p,2} = L_p$, $1 < p < \infty$

Approach 2: Smoothness Morrey spaces

From bmo to spaces of type $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$

$f \in L_1^{\mathrm{loc}}; \quad f_Q = \frac{1}{|Q|} \int_Q f(y)\,\mathrm{d}y, \quad Q \subset \mathbb{R}^n$ cubes

$f \in \mathrm{bmo} \iff$

$$\|f|\mathrm{bmo}\| = \sup_{|Q|\leq 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,\mathrm{d}x + \sup_{|Q|>1} \frac{1}{|Q|} \int_Q |f(x)|\,\mathrm{d}x < \infty$$

$\mathcal{Q}$ ... collection of all dyadic cubes $Q_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$

$Q \in \mathcal{Q} \dashrightarrow$ side-length 2^{-j_Q}

$\{\varphi_j\}_j$ dyadic res. of unity; Frazier/Jawerth (1990):

$$\|f|\mathrm{bmo}\| \sim \sup_{|Q|\leq 1} \frac{1}{|Q|^{1/2}} \left\| \left\| \left(|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)| \right)_{j \geq j_Q} |\ell_2 \right\| \mid L_2(Q) \right\|$$

Approach 2: Smoothness Morrey spaces

The spaces $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$

$0 < p, q \leq \infty$, $\tau \in [0, \infty)$, $s \in \mathbb{R}$, $\{\varphi_j\}_j$ dyadic res. of unity

(i) Besov-type space $B^{s,\tau}_{p,q}$: $f \in \mathcal{S}'$ with

$$\left\| f \mid B^{s,\tau}_{p,q} \right\| = \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\| \left(2^{js} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p(P) \right\| \right)_{j \geq \max\{j_P, 0\}} |\ell_q \right\| < \infty$$

(ii) Triebel-Lizorkin-type space $F^{s,\tau}_{p,q}$: $f \in \mathcal{S}'$ with

$$\left\| f \mid F^{s,\tau}_{p,q} \right\| = \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\| \left\| \left(2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)| \right)_{j \geq \max\{j_P, 0\}} |\ell_q \right\| \mid L_p(P) \right\| < \infty$$

Rem.

- El Baraka ('02), Yuan/Sickel/Yang (2010-)

  W. Yuan, W. Sickel and D. Yang,
  Morrey and Campanato Meet Besov, Lizorkin and Triebel.
  Lecture Notes in Mathematics 2005, Springer, Berlin, 2010.

- $B^{s,0}_{p,q} = B^s_{p,q}$, $F^{s,0}_{p,q} = F^s_{p,q}$, $B^{s,\tau}_{p,q} = F^{s,\tau}_{p,q} = \{0\}$ if $\tau < 0$, elem. properties
- $\mathrm{bmo} = B^{0,1/2}_{2,2} = F^{0,1/p}_{p,2}$, $0 < p < \infty$

Approach 2': Smoothness Morrey spaces

Triebel's Hybrid Spaces

$0 < p, q \leq \infty$, $s \in \mathbb{R}$, $-\frac{n}{p} \leq r < \infty$

global Besov-Morrey and Triebel-Lizorkin-Morrey spaces

$$\|f|L^r B^s_{p,q}\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |Q_{J,M}|^{-(\frac{1}{p}+\frac{r}{n})} \inf_{g\in\mathrm{span}\{\Psi^J_m, m\in\mathbb{Z}^n\}} \|f-g|B^s_{p,q}(2Q_{J,M})\|$$

$$\|f|L^r F^s_{p,q}\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |Q_{J,M}|^{-(\frac{1}{p}+\frac{r}{n})} \inf_{g\in\mathrm{span}\{\Psi^J_m, m\in\mathbb{Z}^n\}} \|f-g|F^s_{p,q}(2Q_{J,M})\|$$

Rem.

- Triebel: local spaces (2013), hybrid spaces (2014)

H. Triebel
Local function spaces, heat and Navier-Stokes equations.
EMS Tracts in Mathematics 20, EMS Publishing House, Zürich, 2013.

H. Triebel
Hybrid Function Spaces, Heat and Navier-Stokes Equations,
EMS Tracts in Mathematics 24, EMS Publishing House, Zürich, 2015.

- $L^r B^s_{p,q} = B^{s,\tau}_{p,q}, \quad L^r F^s_{p,q} = F^{s,\tau}_{p,q}, \qquad \tau = \frac{1}{p} + \frac{r}{n}$

Comparison: Smoothness Morrey spaces

Relation between the different scales

- $\tau \geq \frac{1}{p}$ (with $q = \infty$ if $\tau = \frac{1}{p}$): $B^{s,\tau}_{p,q} = F^{s,\tau}_{p,q} = B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty}$
- $F^{s,\tau}_{p,q} = \mathcal{E}^{s}_{u,p,q}$ with $\tau = \frac{1}{p} - \frac{1}{u}$, $0 < p \leq u < \infty$
- $\mathcal{N}^{s}_{u,p,q} \hookrightarrow B^{s,\tau}_{p,q}$ with $\tau = \frac{1}{p} - \frac{1}{u}$

  coincidence (only) if $\tau = 0$ or $q = \infty$, i.e., $\mathcal{N}^{s}_{u,p,\infty} = B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}$

W. Sickel,
Smoothness spaces related to Morrey spaces – a survey.
Part I: Eurasian Math. J. 3 (2012), 110-149;
Part II: Eurasian Math. J. 4 (2013), 82-124.

Spaces of regular distributions

The problem

convention: $A \in \{B, F\}$, $\mathcal{A} \in \{\mathcal{N}, \mathcal{E}\}$

definition $\curvearrowright$ $A^s_{p,q}, \mathcal{A}^s_{u,p,q}, A^{s,\tau}_{p,q} \subset \mathcal{S}'$ $\dashrightarrow$ tempered distributions

When do the spaces consist of *regular* distributions only, i.e.,

$$A^s_{p,q} \subset L_1^{\mathrm{loc}}, \quad \mathcal{A}^s_{u,p,q} \subset L_1^{\mathrm{loc}}, \quad A^{s,\tau}_{p,q} \subset L_1^{\mathrm{loc}} \quad ?$$

Example Dirac's δ-distribution, singular

$$\delta \in B^s_{p,q} \iff \begin{cases} s < n\left(\frac{1}{p} - 1\right), & \text{or} \\ s = n\left(\frac{1}{p} - 1\right) & \text{and} \quad q = \infty, \end{cases}$$

$$\delta \in F^s_{p,q} \iff s < n\left(\frac{1}{p} - 1\right)$$

$\dashrightarrow$ general criterion?

Spaces of regular distributions

The classical case

$$0 < p \le \infty,\ \sigma_p = n\left(\frac{1}{p} - 1\right)_+,\ 0 < q \le \infty,\ s \in \mathbb{R}$$

$$B^s_{p,q} \subset L_1^{\mathrm{loc}} \iff \begin{cases} s > \sigma_p\ , & \text{or} \\ s = \sigma_p\ , & p > 1 \text{ and } 0 < q \le \min(p,2),\ \text{ or} \\ s = \sigma_p\ , & 0 < p \le 1 \quad \text{and} \quad 0 < q \le 1 \end{cases}$$

$$F^s_{p,q} \subset L_1^{\mathrm{loc}} \iff \begin{cases} s > \sigma_p\ , & \text{or} \\ s = \sigma_p\ , & 0 < p < 1, \quad \text{or} \\ s = \sigma_p\ , & p \ge 1 \quad \text{and} \quad 0 < q \le 2 \end{cases}$$

Rem. Sickel/Triebel (1995); critical smoothness: $s = \sigma_p$

Spaces of regular distributions

Dirac's δ-distribution revisited

Observation $s \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p < u < \infty$, $\tau > 0$

$$\delta \in \mathcal{N}^s_{u,p,q} \iff \begin{cases} s < n\left(\frac{1}{u} - 1\right), & \text{or} \\ s = n\left(\frac{1}{u} - 1\right) & \text{and} \quad q = \infty \end{cases}$$

$$\delta \in \mathcal{E}^s_{u,p,q} \iff s \leq n\left(\frac{1}{u} - 1\right)$$

$$\delta \in B^{s,\tau}_{p,q} \iff s \leq n\left(\frac{1}{p} - \tau - 1\right) \iff \delta \in F^{s,\tau}_{p,q}$$

Rem. H./Moura/Skrzypczak (2016)

Spaces of regular distributions

Smoothness Morrey spaces

basic space

- $\mathcal{M}_{u,p} \subset L_1^{\mathrm{loc}} \iff 1 \le p \le u$ (Piccinini 1969)

smoothness spaces: $s \in \mathbb{R}$, $\tau > 0$, $0 < q \le \infty$, $0 < p < u < \infty$

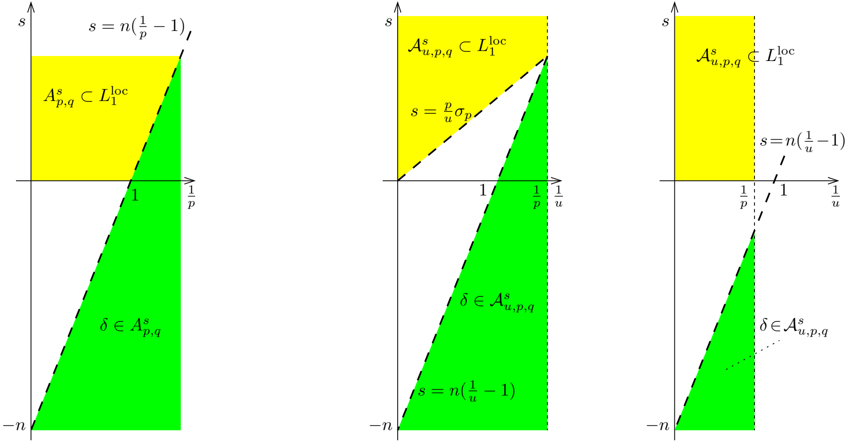
- $A^s_{p,q} \subset L_1^{\mathrm{loc}} \iff s \ge \sigma_p = n\left(\frac{1}{p} - 1\right)_+$
- $\mathcal{A}^s_{u,p,q} \subset L_1^{\mathrm{loc}} \iff s \ge \frac{p}{u}\sigma_p$
- $A^{s,\tau}_{p,q} \subset L_1^{\mathrm{loc}} \iff \begin{cases} s > (1 - p\tau)\frac{n}{p}, & \tau \ge \frac{1}{p} \\ s \ge (1 - p\tau)\sigma_p, & 0 < \tau < \frac{1}{p} \end{cases}$

Rem.

- additional conditions in limiting cases
- H./Moura/Skrzypczak (2016), almost complete result

Spaces of regular distributions

Smoothness Morrey spaces $\mathcal{A}^s_{u,p,q}$



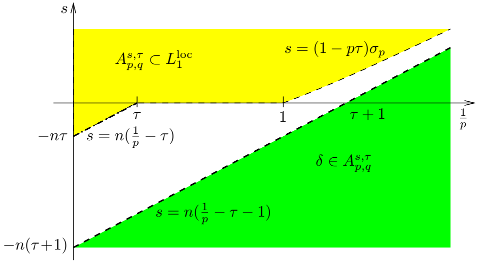
$A^s_{p,q} = \mathcal{A}^s_{p,p,q}$
$0 < p \le u < \infty$

$\mathcal{A}^s_{u,p,q}, \quad 0 < p < 1$
$0 < p < u < \infty$

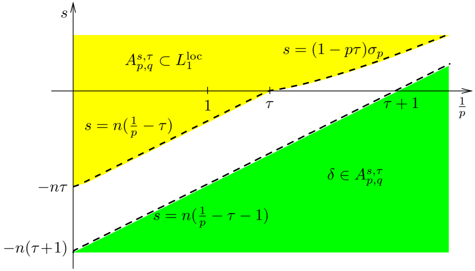
$\mathcal{A}^s_{u,p,q}, \quad p > 1$
$0 < p < u < \infty$

Spaces of regular distributions

Smoothness Morrey spaces $A^{s,\tau}_{p,q}$



$A^{s,\tau}_{p,q}, \quad 0<\tau<1$



$A^{s,\tau}_{p,q}, \quad \tau>1$

Continuous embeddings

Spaces on $\mathbb{R}^n$

Proposition 1

Assume $s_i \in \mathbb{R}$, $0 < p_i \leq u_i < \infty$, $0 < q_i \leq \infty$, $i = 1, 2$.

$$\mathcal{N}^{s_1}_{u_1,p_1,q_1} \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}$$

if, and only if,

$$u_1 \leq u_2, \quad \frac{p_2}{u_2} \leq \frac{p_1}{u_1}, \quad \text{and} \quad \begin{cases} s_1 - \frac{n}{u_1} > s_2 - \frac{n}{u_2}, & \text{or} \\ s_1 - \frac{n}{u_1} = s_2 - \frac{n}{u_2}, & \text{and } q_1 \leq q_2 \ . \end{cases}$$

Rem.

- H./Skrzypczak (2012)
- partial result by Sawano/Sugano/Tanaka (2009)

Continuous embeddings

Spaces on $\mathbb{R}^n$

Proposition 2

Assume $s_i \in \mathbb{R}$, $0 < p_i \le u_i < \infty$, $0 < q_i \le \infty$, $i = 1, 2$.

$$\mathcal{E}^{s_1}_{u_1,p_1,q_1} \hookrightarrow \mathcal{E}^{s_2}_{u_2,p_2,q_2}$$

if, and only if,

$$u_1 \le u_2, \quad \frac{p_2}{u_2} \le \frac{p_1}{u_1}, \quad \text{and} \quad \begin{cases} s_1 - \frac{n}{u_1} > s_2 - \frac{n}{u_2}, & \text{or} \\ s_1 - \frac{n}{u_1} = s_2 - \frac{n}{u_2} & \text{and } u_1 \ne u_2, \quad \text{or} \\ s_1 = s_2, \quad u_1 = u_2 & \text{and } q_1 \le q_2. \end{cases}$$

Rem.

- H./Skrzypczak (2014)
- partial result by Sawano/Sugano/Tanaka (2009)
- parallel results for $A^{s,\tau}_{p,q}$ in Yuan/H./Skrzypczak/Yang (2015)

Continuous embeddings

Further results in limiting cases

- embeddings into 'classical' spaces, like

$$B^{\sigma}_{r,\infty} \hookrightarrow B^{s,\tau}_{p,q} \hookrightarrow B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty} \quad \text{with} \quad \sigma = s + n\big(\tau - \tfrac{1}{p}\big) + \tfrac{n}{r}$$

- embeddings between different scales, in particular

$$F^{s,\tau}_{p,q} = \mathcal{E}^{s}_{u,p,q}, \quad \mathcal{N}^{s}_{u,p,\infty} = B^{s,\tau}_{p,\infty} \quad \text{if} \quad \tau = \tfrac{1}{p} - \tfrac{1}{u},\ 0 < p \leq u < \infty$$

- limiting embeddings of Sobolev and Franke-Jawerth type, like

$$\mathcal{N}^{s_1}_{u_1,p_1,q_1} \hookrightarrow \mathcal{E}^{s}_{u,p,q} \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2} \qquad \text{with} \quad s_i - \tfrac{n}{u_i} \ = \ s - \tfrac{n}{u}$$

[H./Skrzypczak (2014)]

$$B^{s_1,\tau_1}_{p_1,q_1} \hookrightarrow F^{s,\tau}_{p,q} \hookrightarrow B^{s_2,\tau_2}_{p_2,q_2} \qquad \text{with} \quad s_i - \tfrac{n}{p_i} + n\tau_i = s - \tfrac{n}{p} + n\tau$$

[Yuan/H./Moura/Skrzypczak/Yang (2015)]

Continuous embeddings

Embeddings into L_r, $1 \leq r \leq \infty$

Proposition 3

$0 < p < u < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\tau > 0$

- $1 \leq r < \infty$: There is never an embedding $\mathcal{A}^s_{u,p,q} \hookrightarrow L_r$ or $A^{s,\tau}_{p,q} \hookrightarrow L_r$
- $r = \infty$:

$$\mathcal{N}^s_{u,p,q} \hookrightarrow L_\infty \iff \begin{cases} s > \frac{n}{u}, & \text{or} \\ s = \frac{n}{u} & \text{and} \quad q \in (0,1] \end{cases}$$

$$\mathcal{E}^s_{u,p,q} \hookrightarrow L_\infty \iff s > \frac{n}{u}$$

$$A^{s,\tau}_{p,q} \hookrightarrow L_\infty \iff s > n\left(\frac{1}{p} - \tau\right)$$

Rem.

- L_∞ can be replaced by C
- forerunner: Sawano (2009), Sickel (2013)
- H./Skrzypczak (2013, 2014), Yuan/H./Skrzypczak/Yang (2015)

Embeddings into C: Continuity envelopes

The concept

$$\omega(f,t) = \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} |f(x+h)-f(x)| \,, \quad t>0$$

Definition 4

$X \hookrightarrow C$, continuity envelope function $\mathcal{E}_{\mathsf{C}}^X : (0,\infty) \to [0,\infty)$

$$\mathcal{E}_{\mathsf{C}}^X(t) \sim \sup_{\|f|X\|\le 1} \frac{\omega(f,t)}{t}\,, \quad t>0$$

Proposition 5

(i) $X \hookrightarrow \mathrm{Lip}^1 \iff \sup_{t>0} \mathcal{E}_{\mathsf{C}}^X(t) < \infty$

(ii) $X_1 \hookrightarrow X_2 \implies \mathcal{E}_{\mathsf{C}}^{X_1}(t) \le c\, \mathcal{E}_{\mathsf{C}}^{X_2}(t), \quad t>0$

Rem.

- 'fine index' $u_{\mathsf{C}}^X \curvearrowright \mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}_{\mathsf{C}}^X, u_{\mathsf{C}}^X\right)$ **continuity envelope**
- $\mathcal{E}_{\mathsf{C}}^X$ *reasonable* if $X \hookrightarrow C$, *interesting* if $X \not\hookrightarrow \mathrm{Lip}^1$

Embeddings into C: Continuity envelopes

An example

Example If $t \to 0$, then

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}}(t) \sim \begin{cases} |\log t|^{\frac{1}{q'}}, & s = \frac{n}{p}+1, \quad q > 1 \\ t^{-(\frac{n}{p}+1-s)}, & \frac{n}{p} < s < \frac{n}{p}+1 \\ t^{-1}, & s = \frac{n}{p}, \quad 0 < q \le 1 \end{cases}$$

Rem. $\mathfrak{E}_{\mathsf{C}}\left(B^s_{p,q}\right) = \left(\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}}, q\right)$, $\frac{n}{p} < s \le \frac{n}{p}+1$, $\mathfrak{E}_{\mathsf{C}}\left(B^{n/p}_{p,q}\right) = \left(t^{-1}, \infty\right)$

parallel results for F-spaces

Embeddings into C: Continuity envelopes

Smoothness Morrey spaces

Theorem 6

(i) Let $0 < p < u < \infty$. If $\mathcal{N}^s_{u,p,q} \hookrightarrow C$ and $\mathcal{N}^s_{u,p,q} \not\hookrightarrow \mathrm{Lip}^1$, then

$$\mathcal{E}_{\mathsf{C}}^{\mathcal{N}^s_{u,p,q}}(t) \sim \mathcal{E}_{\mathsf{C}}^{B^s_{u,q}}(t), \quad t \to 0$$

(ii) Let $\tau > 0$. If $t \to 0$, then

$$\mathcal{E}_{\mathsf{C}}^{A^{s,\tau}_{p,q}}(t) \sim \begin{cases} t^{s+n(\tau-\frac{1}{p})-1}, & 0 < s - n\left(\frac{1}{p} - \tau\right) < 1 \\ |\log t|, & s - n\left(\frac{1}{p} - \tau\right) = 1 \end{cases}$$

Rem.

- $\frac{n}{u} \leq s \leq \frac{n}{u} + 1$ (with additional assumptions in limiting cases)
- similar results for spaces $\mathcal{E}^s_{u,p,q}$, slightly different from $F^s_{u,q}$
- Yuan/H./Moura/Skrzypczak/Yang (2015)

Embeddings into L_∞: Growth envelopes

The concept

$$f^*(t) = \inf \{ s \geq 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \leq t \}, \quad t \geq 0$$

Definition 7

$X \subset L_1^{\mathrm{loc}}$, ***growth envelope function*** $\mathcal{E}_G^X : (0,\infty) \to [0,\infty]$

$$\mathcal{E}_G^X(t) \sim \sup_{\|f|X\| \leq 1} f^*(t), \quad t > 0$$

Proposition 8

(i) $X \hookrightarrow L_\infty \iff \sup_{t>0} \mathcal{E}_G^X(t) < \infty$

(ii) $X_1 \hookrightarrow X_2 \implies \mathcal{E}_G^{X_1}(t) \leq c\, \mathcal{E}_G^{X_2}(t), \quad t > 0$

(iii) X rearrangement-invariant with fundamental function φ_X

$$\mathcal{E}_G^X(t) \sim \frac{1}{\varphi_X(t)}, \quad t > 0$$

Rem.

- 'fine index' $u_G^X \curvearrowright \mathfrak{E}_G(X) = (\mathcal{E}_G^X, u_G^X)$ 'growth envelope'
- $\mathcal{E}_G^X$ *reasonable* for $X \subset L_1^{\mathrm{loc}}$, *interesting* for $X \not\hookrightarrow L_\infty$

Growth envelopes in Smoothness Morrey spaces

Examples

Example $\mathfrak{E}_{\mathsf{G}}(L_{p,q}) = \left(t^{-\frac{1}{p}}, q\right), \quad 0<p<\infty,\ 0<q\leq\infty$

Example $0<p<\infty,\ 0<q\leq\infty,\ \ s>\sigma_p=n\left(\frac{1}{p}-1\right)_+$

(i) For $s<\frac{n}{p}$, $\quad \mathcal{E}_{\mathsf{G}}^{A^s_{p,q}}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}}, \quad t\to 0$

(ii) If $s=\frac{n}{p}$, $1<q\leq\infty$, then

$$\mathcal{E}_{\mathsf{G}}^{B^{n/p}_{p,q}}(t) \sim |\log t|^{\frac{1}{q'}}, \quad t\to 0$$

(iii) We have $\quad \mathcal{E}_{\mathsf{G}}^{A^s_{p,q}}(t) \sim t^{-\frac{1}{p}}, \quad t\to\infty$

Rem. $\mathfrak{E}_{\mathsf{G}}\left(B^s_{p,q}\right) = \left(\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}, q\right), \quad \mathfrak{E}_{\mathsf{G}}\left(F^s_{p,q}\right) = \left(\mathcal{E}_{\mathsf{G}}^{F^s_{p,q}}, p\right)$

Embeddings into L_∞: Growth envelopes in Morrey spaces

The bottom line

Proposition 9

$0 < p < u < \infty$

$$\mathcal{E}_{\mathsf{G}}^{\mathcal{M}_{u,p}(\mathbb{R}^n)}(t) = \infty, \quad t > 0$$

Rem.

- $0 < p = u < \infty$: $\mathcal{E}_{\mathsf{G}}^{L_p(\mathbb{R}^n)}(t) \sim t^{-1/p}$, $t > 0$
- H./Moura (2016), idea: consider

$$f(x) = |x|^{-\frac{n}{u}} \begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$g(x) = \sum_{k=0}^{\infty} f(x - m_k), \quad m_k = (2^k, 0, \cdots, 0)$$

$\curvearrowright$ $g \in \mathcal{M}_{u,p}$, $g^*(t) = \infty$, $t > 0$

- if $\Omega \subset \mathbb{R}^n$ bounded
  $\curvearrowright$ $\mathcal{M}_{u,p}(\Omega) \hookrightarrow L_p(\Omega) \curvearrowright \mathcal{E}_{\mathsf{G}}^{\mathcal{M}_{u,p}(\Omega)}(t) \leq ct^{-1/p} < \infty$

Growth envelopes

Smoothness Morrey spaces

Theorem 10

- Let $0 < p < u < \infty$, $\mathcal{A}^s_{u,p,q} \subset L_1^{\mathrm{loc}}$ and $\mathcal{A}^s_{u,p,q} \not\hookrightarrow L_\infty$. Then

$$\mathcal{E}_{\mathsf{G}}^{\mathcal{A}^s_{u,p,q}(\mathbb{R}^n)}(t) = \infty, \quad t > 0.$$

- Let $\tau > 0$, $A^{s,\tau}_{p,q} \subset L_1^{\mathrm{loc}}$ and $A^{s,\tau}_{p,q} \not\hookrightarrow L_\infty$. Then

$$\mathcal{E}_{\mathsf{G}}^{A^{s,\tau}_{p,q}(\mathbb{R}^n)}(t) = \infty, \quad t > 0$$

Rem.

- $0 < p = u < \infty$, $\tau = 0$: $\mathcal{E}_{\mathsf{G}}^{A^{s,\tau}_{p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{\mathcal{A}^s_{u,p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{A^s_{p,q}}(t) < \infty$
- essential again: $\mathbb{R}^n$
- $\frac{p}{u}\sigma_p \le s \le \frac{n}{u}$, $\tau = \frac{1}{p}$ (with additional conditions, including bmo)
- H./Moura (2016), H./Moura/Skrzypczak (2016)
  idea: atomic decomposition (with moment conditions)

Smoothness Morrey spaces on domains

Definition

$\Omega \subset \mathbb{R}^n$ bounded C^∞ domain

Definition 11

$0 < p \le u < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$

$$A^s_{p,q}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega) : \ \exists\, g \in A^s_{p,q}(\mathbb{R}^n) : \ f = g|_\Omega \right\}$$

$$\text{with} \quad \left\| f \,|A^s_{p,q}(\Omega) \right\| = \inf_{f=g|_\Omega} \left\| g \,|A^s_{p,q}(\mathbb{R}^n) \right\|$$

$$\mathcal{A}^s_{u,p,q}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega) : \ \exists\, g \in \mathcal{A}^s_{u,p,q}(\mathbb{R}^n) : \ f = g|_\Omega \right\}$$

$$\text{with} \quad \left\| f \,|\mathcal{A}^s_{u,p,q}(\Omega) \right\| = \inf_{f=g|_\Omega} \left\| g \,|\mathcal{A}^s_{u,p,q}(\mathbb{R}^n) \right\|$$

Rem. Sawano (2010): $N \in \mathbb{N}$, $N^{-1} \le p \le u < \infty$, $|s| < N$, $N^{-1} < q \le \infty$
for any N there is a common extension operator ext_N,

$$\mathrm{ext}_N : \mathcal{N}^s_{u,p,q}(\Omega) \to \mathcal{N}^s_{u,p,q}(\mathbb{R}^n) \quad \text{with} \quad \mathrm{re} \circ \mathrm{ext}_N = \mathrm{id}_{\mathcal{N}^s_{u,p,q}(\Omega)}$$

Smoothness Morrey spaces on domains

Compact embeddings: the 'classical' situation

Let $s_i \in \mathbb{R}$, $0 < q_i \leq \infty$, $0 < p_i \leq \infty$, $i = 1, 2$.

$$\mathrm{id}^\Omega : A^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2,q_2}(\Omega)$$

is compact if, and only if, $\dfrac{s_1 - s_2}{n} > \max\left\{0, \dfrac{1}{p_1} - \dfrac{1}{p_2}\right\}$,

$$e_k(\mathrm{id}^\Omega) \sim k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}$$

Rem. dyadic entropy numbers of $T \in \mathcal{L}(X, Y)$

$$e_k(T) = \inf\left\{\varepsilon > 0 : T(U_X) \subset \bigcup_{l=1}^{2^{k-1}} \{y_l + \varepsilon U_Y\}\right\}$$

Smoothness Morrey spaces on domains

Compact embeddings

Theorem 12

Let $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$, $0 < p_i \le u_i < \infty$, $i = 1, 2$.

$$\mathrm{id}^{\Omega} : \mathcal{A}^{s_1}_{u_1,p_1,q_1}(\Omega) \hookrightarrow \mathcal{A}^{s_2}_{u_2,p_2,q_2}(\Omega)$$

is compact if, and only if,

$$\frac{s_1 - s_2}{n} > \max\left\{0, \frac{1}{u_1} - \frac{1}{u_2}, \frac{p_1}{u_1}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right\}.$$

Rem. H./Skrzypczak 2013: (complete) characterisation of the continuity of id^{Ω}
partial forerunner: Dzhumakaeva (1985), Netrusov (1984)

Corollary 13

If $\frac{s_1 - s_2}{n} > \max\left\{0, \frac{1}{p_1} - \frac{1}{u_2}, \frac{1}{u_1} - \frac{1}{p_2}\right\}$, then $e_k(\mathrm{id}^{\Omega}) \sim k^{-\frac{s_1 - s_2}{n}}$, $\quad k \in \mathbb{N}$.

Smoothness Morrey spaces on domains

Compact embeddings: approximation numbers

$\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $\mathcal{A}^s_{u,p,q}(\Omega)$ defined by restriction

approximation numbers of $T \in \mathcal{L}(X, Y)$

$$a_k(T) = \inf\{\|T - S\| : S \in \mathcal{L}(X, Y), \mathrm{rank}\, S < k\}, \quad k \in \mathbb{N}$$

Example $p \in [2, \infty], q \in (0, \infty], s > \frac{n}{p}$

$$a_k\left(\mathrm{id}^\Omega : A^s_{p,q}(\Omega) \hookrightarrow C(\Omega)\right) \sim k^{-\frac{s}{n}+\frac{1}{p}}, \quad k \in \mathbb{N}$$

Corollary 14

$u \in [2, \infty)$, $p \in (0, u]$, $q \in (0, \infty]$, $\frac{n}{u} < s < \frac{n}{u} + 1$

$$a_k\left(\mathrm{id}^\Omega : \mathcal{A}^s_{u,p,q}(\Omega) \to C(\Omega)\right) \sim k^{-\frac{s}{n}+\frac{1}{u}}, \quad k \in \mathbb{N}$$

Rem. $a_k\left(\mathrm{id} : X(\Omega) \longrightarrow C(\Omega)\right) \leq c\, k^{-\frac{1}{n}}\, \mathcal{E}^X_C\left(k^{-\frac{1}{n}}\right), \quad k \in \mathbb{N}$

Weighted Morrey spaces

Preparation: Weights

Muckenhoupt $\mathcal{A}_p$ weights: $w \in L_1^{\mathrm{loc}}$, positive

- $w \in \mathcal{A}_p,\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1$:

$$\left(\frac{1}{|B|} \int_B w(x)\,\mathrm{d}x \right)^{1/p} \left(\frac{1}{|B|} \int_B w(x)^{-p'/p}\,\mathrm{d}x \right)^{1/p'} \leq A$$

- $w \in \mathcal{A}_1$: $\quad Mw(x) \leq Aw(x)$ for a.e. $x \in \mathbb{R}^n$
- $\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p$
- $r_w = \inf\{r \geq 1 : w \in \mathcal{A}_r\}$
- $\boldsymbol{S}_{\mathrm{sing}}(w) = \left\{ x_0 : \inf_{Q_{\nu,\boldsymbol{m}} \ni x_0} \frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} = 0 \right\} \cup \left\{ x_0 : \sup_{Q_{\nu,\boldsymbol{m}} \ni x_0} \frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} = \infty \right\}$

  set of (local) singularities for $w \in \mathcal{A}_\infty$

Notation: $w(\Omega) = \int_\Omega w(x)\,\mathrm{d}x$, $\quad Q_{\nu,m}$ dyadic cubes, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$

Weighted Morrey spaces

Example weights

- $w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha, & \text{if } |x| \leq 1 \\ |x|^\beta, & \text{if } |x| > 1 \end{cases}$

$$w_{\alpha,\beta} \in \mathcal{A}_p \iff \begin{cases} -n < \alpha, \beta < n(p-1), & \text{if } p > 1 \\ -n < \alpha, \beta \leq 0, & \text{if } p = 1 \end{cases}$$

$$\curvearrowright \quad r_{w_{\alpha,\beta}} = 1 + \frac{\max(\alpha,\beta,0)}{n}, \qquad \boldsymbol{S}_{\text{sing}}(w_{\alpha,\beta}) = \begin{cases} \{0\}, & \alpha \neq 0 \\ \emptyset, & \alpha = 0 \end{cases}$$

- $w_n(x) = \begin{cases} |x_n|^\alpha, & \text{if } |x_n| \leq 1 \\ 1, & \text{otherwise} \end{cases} \in \mathcal{A}_p \iff -1 < \alpha < p-1$

$$\curvearrowright \quad r_{w_n} = 1 + \max(\alpha, 0), \qquad \boldsymbol{S}_{\text{sing}}(w_n) = \begin{cases} \mathbb{R}^{n-1}, & \alpha \neq 0 \\ \emptyset, & \alpha = 0 \end{cases}$$

Weighted Morrey spaces

Local singularities of weights

$$\boldsymbol{S}_{\rm sing}(w) = \left\{x_0 : \inf_{Q_{\nu,\boldsymbol{m}}\ni x_0} \frac{w(Q_{\nu,\boldsymbol{m}})}{|Q_{\nu,\boldsymbol{m}}|} = 0\right\} \cup \left\{x_0 : \sup_{Q_{\nu,\boldsymbol{m}}\ni x_0} \frac{w(Q_{\nu,\boldsymbol{m}})}{|Q_{\nu,\boldsymbol{m}}|} = \infty\right\}$$

Observation

- $\boldsymbol{S}_{\rm sing}(w)$ not necessarily bounded, $\boldsymbol{S}_{\rm sing}(w)$ closed?
- H./Skrzypczak (2012): $|\boldsymbol{S}_{\rm sing}(w)| = 0$

Proposition 15

Let $w \in \mathcal{A}_\infty$. Then $|\overline{\boldsymbol{S}_{\rm sing}(w)}| = 0$.

Rem. $\boldsymbol{S}_{\rm sing}(w)$ not necessarily closed

Corollary 16

Let $w \in \mathcal{A}_\infty$. Then any cube $Q \subset \mathbb{R}^n$ contains a regularity cube $\widetilde{Q} \subset Q$, that is, where $w(x) \sim c$, $x \in \widetilde{Q}$.

Weighted spaces of Morrey type

Definition

$0 < p \le u < \infty$, Morrey space $\mathcal{M}_{u,p}$

$$\|f|\mathcal{M}_{u,p}\| = \sup_{x\in\mathbb{R}^n,\ R>0} R^{\frac{n}{u}-\frac{n}{p}} \left(\int_{B(x,R)} |f(y)|^p \,\mathrm{d}y\right)^{1/p}$$

$$\sim \sup_{B \text{ ball}} |B|^{\frac{1}{u}-\frac{1}{p}} \left(\int_B |f(y)|^p \,\mathrm{d}y\right)^{1/p} < \infty$$

$0 < p \le u < \infty$, $w \in \mathcal{A}_\infty$, weighted Morrey space $\mathcal{M}_{u,p}(w)$:

$$\|f|\mathcal{M}_{u,p}(w)\| = \sup_{B \text{ ball}} w(B)^{\frac{1}{u}-\frac{1}{p}} \left(\int_B |f(y)|^p w(y) \,\mathrm{d}y\right)^{1/p} < \infty$$

Rem.

- $\mathcal{M}_{u,p}(1) = \mathcal{M}_{u,p}$, Komori/Shirai (2009), Izuki/Sawano/Tanaka (2010)
- different approaches, e.g. Mustafayev/Ünver (2015), ...
- $\|f|L_p(w)\| = \|fw^{1/p}|L_p\|$, but for $p < u$:

  $\|f|\mathcal{M}_{u,p}(w)\| \sim \|fw^{1/p}|\mathcal{M}_{u,p}\| \iff w \sim \mathrm{const}, \ \boldsymbol{S}_{\mathrm{sing}}(w) = \emptyset$

Embeddings from Morrey to Morrey spaces

Power-type weights

Assume $\alpha, \beta > -n$ and $\quad w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \text{if } |x| \le 1, \\ |x|^\beta & \text{if } |x| > 1. \end{cases}$

Theorem 17

Let $0 < u_1, u_2 < \infty$, $0 < p_1 \le u_1$, $0 < p_2 < u_2$. Then

$$\mathcal{M}_{u_1,p_1}(w_{\alpha,\beta}) \hookrightarrow \mathcal{M}_{u_2,p_2}$$

if, and only if,

$$u_1 \ge u_2, \quad p_1 \ge p_2 \quad \text{and} \quad \begin{cases} -n < \alpha \le 0 \le \beta, & \text{if } u_1 = u_2, \\ 1 + \frac{\alpha}{n} \le \frac{u_1}{u_2} \le 1 + \frac{\beta}{n}, & \text{if } u_1 > u_2. \end{cases}$$

Rem.

- $\alpha = \beta$: $\mathcal{M}_{u_1,p_1}(|x|^\alpha) \hookrightarrow \mathcal{M}_{u_2,p_2} \iff u_1 \ge u_2, p_1 \ge p_2, \alpha = n(\frac{u_1}{u_2} - 1)$
- unweighted case, $\alpha = \beta = 0$

  $\mathcal{M}_{u_1,p_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{u_2,p_2}(\mathbb{R}^n) \iff p_2 \le p_1 \le u_1 = u_2$ (Rosenthal 2009)

  $\mathcal{M}_{u_1,p_1}(Q) \hookrightarrow \mathcal{M}_{u_2,p_2}(Q) \iff p_2 \le p_1, \quad u_2 \le u_1$ (Piccinini 1969)

Embeddings from Morrey to Morrey spaces

General $\mathcal{A}_\infty$ weights: Necessary conditions

Proposition 18

Let $0 < p_i \leq u_i < \infty$, $i = 1, 2$ and $w \in \mathcal{A}_\infty$. Then

$$\mathrm{id} : \mathcal{M}_{u_1,p_1}(w) \hookrightarrow \mathcal{M}_{u_2,p_2}$$

implies that

- $u_1 \geq u_2$,
- $p_1 \geq p_2$,
- $\sup_Q |Q|^{\frac{1}{u_2}} w(Q)^{-\frac{1}{u_1}} < \infty$, and
- $r_w \geq \frac{u_1}{u_2} \geq 1$.

Example $w \equiv 1$: $\mathcal{M}_{u_1,p_1} \hookrightarrow \mathcal{M}_{u_2,p_2} \iff p_2 \leq p_1 \leq u_1 = u_2$

Embeddings from Morrey to Morrey spaces

General $\mathcal{A}_\infty$ weights: Sufficient conditions

Proposition 19

Let $0 < p_i \leq u_i < \infty$, $i = 1, 2$ and $w \in \mathcal{A}_\infty$. Assume

- $w \in \mathcal{A}_{p_1/p_2}$, that is, $r_w < \frac{p_1}{p_2}$, and
- $\sup_Q |Q|^{\frac{1}{u_2}} w(Q)^{-\frac{1}{u_1}} < \infty$,

then

$$\mathrm{id} : \mathcal{M}_{u_1,p_1}(w) \hookrightarrow \mathcal{M}_{u_2,p_2}.$$

Example $w = w_{\alpha,\beta}$, $\alpha, \beta > -n$

$$\frac{p_1}{p_2} > r_{w_{\alpha,\beta}} = 1 + \frac{\max(0,\alpha,\beta)}{n} \geq 1 + \frac{\max(0,\alpha)}{n} \geq 1$$

$\curvearrowright$ more restrictive than needed

Embeddings between weighted Morrey spaces

$\mathcal{A}_1$ weights

Theorem 20

Let $0 < p_i \le u_i < \infty$, $w \in \mathcal{A}_1$. Then

$$\mathrm{id} : \mathcal{M}_{u_1,p_1}(w) \hookrightarrow \mathcal{M}_{u_2,p_2}$$

if, and only if,

$$u_1 = u_2, \qquad p_2 \le p_1 \qquad \text{and} \qquad \inf_{Q:\,|Q|=1} w(Q) > 0.$$

Example $w \equiv 1$: $\mathcal{M}_{u_1,p_1} \hookrightarrow \mathcal{M}_{u_2,p_2} \iff p_2 \le p_1 \le u_1 = u_2$,

$$\text{since} \quad \inf_{Q:\,|Q|=1} w(Q) = \inf_{Q:\,|Q|=1} |Q| = 1 > 0$$

Embeddings between weighted Morrey spaces

Compact embeddings

Corollary 21

Let $0 < p_i \le u_i < \infty$, $i = 1, 2$, $w \in \mathcal{A}_\infty$. Whenever there is a continuous embedding,

$$\mathcal{M}_{u_1,p_1}(w) \hookrightarrow \mathcal{M}_{u_2,p_2},$$

it can never be compact, in particular, a continuous embedding of type

$$L_{u_1}(w) \hookrightarrow L_{u_2}$$

can never be compact for $w \in \mathcal{A}_\infty$.

Rem. reason: existence of *regularity cubes* for $w \in \mathcal{A}_\infty$ and non-compactness of $\mathcal{M}_{u_1,p_1}(Q) \hookrightarrow \mathcal{M}_{u_2,p_2}(Q)$ or even $L_{u_1}(Q) \hookrightarrow L_{u_2}(Q)$

Way out $\dashrightarrow$ shrink source space, involve additional smoothness

Weighted embeddings of Smoothness Morrey spaces

An example

Proposition 22

$s_i \in \mathbb{R}$, $0 < q_i \leq \infty$, $i = 1, 2$, $0 < u_1 < \infty$, $0 < p_2 < u_2 < \infty$, $\alpha > -n$

$$\mathrm{id}^{\alpha}_{B\mathcal{N}} : B^{s_1}_{u_1,q_1}(|x|^{\alpha}) \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}$$

if, and only if,

$$\frac{\alpha}{u_1} \geq n\left(\frac{1}{u_2} - \frac{1}{u_1}\right)_+ \quad \text{and} \quad \begin{cases} s_1 - s_2 > \frac{\alpha}{u_1} - n\left(\frac{1}{u_2} - \frac{1}{u_1}\right), & q_1 > q_2, \\ s_1 - s_2 \geq \frac{\alpha}{u_1} - n\left(\frac{1}{u_2} - \frac{1}{u_1}\right), & q_1 \leq q_2. \end{cases}$$

$\mathrm{id}^{\alpha}_{B\mathcal{N}}$ is compact if, and only if, $s_1 - \frac{n}{u_1} - s_2 + \frac{n}{u_2} > \frac{\alpha}{u_1} > n\left(\frac{1}{u_2} - \frac{1}{u_1}\right)_+$

Rem. $B^{s_1}_{p_1,q_1}(|x|^{\alpha}) \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}$ compact $\iff$ $B^{s_1}_{p_1,q_1}(|x|^{\alpha}) \hookrightarrow B^{s_2}_{p_2,q_2}$ compact

Weighted embeddings of Smoothness Morrey spaces

The general case

Some open questions

- general criterion for continuity of $B^{s_1}_{p_1,q_1}(w) \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}$ ✓
- general criterion for compactness of $B^{s_1}_{p_1,q_1}(w) \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}$?
- continuity and compactness of

$$\mathcal{A}^{s_1}_{u_1,p_1,q_1}(w) \hookrightarrow \mathcal{A}^{s_2}_{u_2,p_2,q_2}, \quad \mathcal{A}^{s_1}_{u_1,p_1,q_1}(w_1) \hookrightarrow \mathcal{A}^{s_2}_{u_2,p_2,q_2}(w_2) \quad ?$$

- influence of Morrey parameters p_i for compactness ?
- interplay with the weight ?
- entropy numbers, approximation numbers ?

Thank you very much for your attention,

and . . .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