ANISOTROPIC HARDY-LORENTZ SPACES WITH VARIABLE EXPONENTS

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If X is a function space, the Hardy space $H(\mathbb{R}^n, X)$ on $\mathbb{R}^n$ modeled on X consists of all those tempered distributions f on $\mathbb{R}^n$ such that the maximal function $\mathcal{M}(f)$ of f is in X.

Bownik (2003)

Bownik studied anisotropic Hardy spaces on $\mathbb{R}^n$. If A is a dilation in $\mathbb{R}^n$, we define

$$\mathcal{M}_A(f) = \sup_{k \in \mathbb{Z}} |f * \varphi_{A,k}|, \quad f \in S'(\mathbb{R}^n).$$

Bownik characterizes anisotropic Hardy spaces by maximal functions like $\mathcal{M}_A$. Recently, Liu, Yang, and Yuan (2015) have extended Bownik's results by studying anisotropic Hardy spaces on $\mathbb{R}^n$ modeled on Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.

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Ephremidze, Kokilashvili and Samko (2008)

Ephremidze, Kokilashvili and Samko introduced variable exponent Lorentz Spaces $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Anisotropic context. M. Bownik, "Anisotropic Hardy Spaces and Wavelets", Mem. Amer. Math. Soc., 164 (2003)

The anisotropic geometry

A is a $n \times n$ real matrix such that $\min_{\lambda \in \sigma(A)} |\lambda| > 1$.

- $\Delta = \{x \in \mathbb{R}^n : \ |Px| < 1\}$ is called the ellipsoid generated by P, $det(P) \neq 0$.
- For every $k \in \mathbb{Z}$, $B_k = A^k \Delta$. $|B_k| = b^k$, where $b = |det\ A|$.
- The step quasinorm $\rho_A(x) = \begin{cases} b^k, & x \in B_{k+1} \setminus B_k, \ k \in \mathbb{Z}, \\ 0, & x = 0. \end{cases}$

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Anisotropic maximal functions

$\varphi_k(x) = |det\ A|^{-k} \varphi(A^{-k}x)$ and $S_N = \{\varphi \in S(\mathbb{R}^n) : \ \sup_{x \in \mathbb{R}^n} \rho_A(x)^N |D^\alpha \varphi(x)| \leq 1, \ \alpha \in \mathbb{N}^n$ and $s(\alpha) \leq N\}$, $s(\alpha) = \alpha_1 + ... + \alpha_n$.

- $M^0_\varphi(f)(x) = \sup_{k \in \mathbb{Z}} |(f * \varphi_k)(x)|$ **(Radial maximal function).**
- $M_\varphi(f)(x) = \sup_{k \in \mathbb{Z}\, ,\, y \in x + B_k} |(f * \varphi_k)(y)|$ **(Nontangential maximal function).**
- $M^0_N(f) = \sup_{\varphi \in S_N} M^0_\varphi(f)$ **(Radial grandmaximal function).**
- $M_N(f) = \sup_{\varphi \in S_N} M_\varphi(f)$ **(Nontangential grandmaximal function).**

Variable exponents Lebesgue spaces. L. Diening et al, "Lebesgue and Sobolev Spaces with Variable Exponents ", vol. 2017 of Lectures Notes in Mathematics, Springer, Heilderberg, 2011.

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- $p : \Omega \to (0, \infty)$ a measurable function.
- $0 \leq p_-(\Omega) \leq p_+(\Omega) < \infty$, where $p_-(\Omega) = \inf_{x \in \Omega} p(x)$ and $p_+(\Omega) = \sup_{x \in \Omega} p(x)$.

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- $\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 \ : \ \int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}, \quad f \in L^{p(\cdot)}(\Omega).$

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- By $\mathbb{P}$ we represent the class of bounded measurable functions p such that there exist the limits $p(0) =: \lim_{t\to 0^+} p(t)$ and $p(\infty) =: \lim_{t\to+\infty} p(t)$ and the following conditions are satisfied

$$|p(t) - p(0)| \leq \frac{C}{|\ln t|}, \text{ for } 0 < t \leq 1/2$$

and .

$$|p(t) - p(\infty)| \leq \frac{C}{\ln(e+t)}, \text{ for } t \in (0,\infty).$$

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- $\mathbb{P}_a = \mathbb{P} \cap \mathfrak{P}_a$.

Classical Lorentz spaces $L^{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$

- $\mu_f(s) = |\{x \in \mathbb{R}^n : \ |f(x)| > s\}|, \quad s \geq 0.$
- $f^*(t) = \inf\{s \geq 0 : \ \mu_f(s) \leq t\}, \quad t \in [0, \infty).$
- $\|f\|_{L^{p,q}(\mathbb{R}^n)} = \begin{cases} \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} & \text{if } q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$

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$\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ Ephremidze, Kokilashvili and Samko (2008)

- $\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t)\|_{L^{q(\cdot)}(0,\infty)}, \quad p, q \in \mathfrak{P}_0.$
- $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \ t \in (0, \infty).$
- $\|f\|^{(1)}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^{**}(t)\|_{L^{q(\cdot)}(0,\infty)}, \quad f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n).$
- $\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sim \|f\|^{(1)}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$, $p \in \mathbb{P}_0$, $q \in \mathbb{P}_1$, $p(0) > 1$, and $p(\infty) > 1$

$L_{p(\cdot),q}(\mathbb{R}^n)$ Kempa and Vybiral (2014)

- $\left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \sim p^{1/q}\left(\sum_{-\infty}^{\infty} \|2^k \chi_{\{x\in\mathbb{R}^n:|f(x)|>2^k\}}\|_{L^p(\mathbb{R}^n)}^q\right)^{1/q}.$
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$L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ Kempa and Vybiral (2014)

- $\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(f_k) = \sum_k \inf\left\{\lambda_k > 0 \;:\; \rho_{p(\cdot)}\left(f_k/\lambda_k^{1/q(\cdot)}\right) \le 1\right\}.$
- $\|f_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\{\mu > 0 \;:\; \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(f_k/\mu) \le 1\}.$
- $\|f\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \|(2^k \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>2^k\}})_{k=-\infty}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$

$\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$	$L_{p(\cdot),q}(\mathbb{R}^n)$	$L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$
$p(\cdot) = p, \;\; q(\cdot) = q$ constants $\mathcal{L}^{p(\cdot),q(\cdot)} = L^{p,q}$	$p(\cdot) = p$ constant $L_{p(\cdot),q} = L^{p,q}$	$p(\cdot) = p, \;\; q(\cdot) = q$ constants $L_{p(\cdot),q(\cdot)} = L^{p,q}$
$p(\cdot) = q(\cdot)$ $\mathcal{L}^{p(\cdot),q(\cdot)} \neq L^{p(\cdot)}$		$p(\cdot) = q(\cdot)$ $L_{p(\cdot),q(\cdot)} = L^{p(\cdot)}$
$[\mathcal{L}^{p(\cdot),q(\cdot)}]' = \mathcal{L}^{p'(\cdot),q'(\cdot)}$	$[L_{p(\cdot),q}]' = ?$	$[L_{p(\cdot),q(\cdot)}]' = ?$

$\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$	$L_{p(\cdot),q}(\mathbb{R}^n)$	$L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$
$p(\cdot)=p,\ \ q(\cdot)=q$ constants $\mathcal{L}^{p(\cdot),q(\cdot)}=L^{p,q}$	$p(\cdot)=p$ constant $L_{p(\cdot),q}=L^{p,q}$	$p(\cdot)=p,\ \ q(\cdot)=q$ constants $L_{p(\cdot),q(\cdot)}=L^{p,q}$
$p(\cdot)=q(\cdot)$ $\mathcal{L}^{p(\cdot),q(\cdot)}\neq L^{p(\cdot)}$		$p(\cdot)=q(\cdot)$ $L_{p(\cdot),q(\cdot)}=L^{p(\cdot)}$
$[\mathcal{L}^{p(\cdot),q(\cdot)}]'=\mathcal{L}^{p'(\cdot),q'(\cdot)}$	$[L_{p(\cdot),q}]'=?$	$[L_{p(\cdot),q(\cdot)}]'=?$

We use extrapolation arguments in some of our proof, that implies that the Hardy-Littlewood Maximal function boundedness on the dual space of the variable exponents Lorentz space is needed, so we chose $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to define our Hardy-Lorentz space.

$H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

Let $N \in \mathbb{N}$ and $p, q \in \mathfrak{P}_0$. The $(p(\cdot), q(\cdot))$-anisotropic Hardy-Lorentz space $H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ associated with A is the set of all those $f \in S'(\mathbb{R}^n)$ such that $M_N(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. On $H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ we consider the quasinorm $\| \ \|_{H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}$ defined by

$$\|f\|_{H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} = \|M_N(f)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}, \quad f \in H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A).$$

Atoms

Let $1 < r \leq \infty$, $s \in \mathbb{N}$ and $p, q \in \mathfrak{P}_0$. We say that a measurable function a on $\mathbb{R}^n$ is a $(p(\cdot), q(\cdot), r, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ when a satisfies

(a) supp $a \subseteq x_0 + B_k$.

(b) $\|a\|_r \leq b^{k/r} \|\chi_{x_0+B_k}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$.

(c) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$, for every $\alpha \in \mathbb{N}^n$ such that $s(\alpha) \leq s$.

Here, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

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Here, if $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

We define the atomic $(p(\cdot), q(\cdot))$-anisotropic Hardy-Lorentz space $H^{p(\cdot),q(\cdot)}_{r,s}(\mathbb{R}^n, A)$ as the set of all distributions $f \in S'(\mathbb{R}^n)$ such that:

1. $\displaystyle f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$.
2. $\displaystyle \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

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$$\|f\|_{H^{p(\cdot),q(\cdot)}_{r,s}(\mathbb{R}^n,A)} = \inf \left\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1

Let $f \in S'(\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^n)$ such that $\int \varphi \neq 0$. Assume that $p, q \in \mathbb{P}_0$. Then, the following assertions are equivalent.

(i) There exists $N_0 \in \mathbb{N}$ such that, for every $N \geq N_0$, $f \in H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

(ii) $M_\varphi(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(iii) $M_\varphi^0(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Moreover, for every $g \in S'(\mathbb{R}^n)$ the quantities $\|M_N(g)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$, $N \geq N_0$, $\|M_\varphi^0(g)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ and $\|M_\varphi(g)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ are equivalent.

Theorem 2

Let $p, q \in \mathbb{P}_0$.

(i) There exist $s_0 \in \mathbb{N}$ and $C > 0$ such that if, for every $j \in \mathbb{N}$, $\lambda_j > 0$ and a_j is a $(p(\cdot), q(\cdot), \infty, s_0)$-atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, satisfying that $\sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then $f = \sum_{j\in\mathbb{N}} \lambda_j a_j \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and

$$\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \leq C \left\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

If also $p(0) \leq q(0)$, then there exists $r_0 > 1$ such that for every $r_0 < r < \infty$ the above assertion is true when $(p(\cdot), q(\cdot), \infty, s_0)$-atoms are replaced by $(p(\cdot), q(\cdot), r, s_0)$-atoms.

Theorem 2

(ii) There exists $s_0 \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s \geq s_0$, and $1 < r \leq \infty$, we can find $C > 0$ such that, for every $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, there exist, for each $j \in \mathbb{N}$, $\lambda_j > 0$ and a $(p(\cdot), q(\cdot), r, s)$-atom a_j associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, satisfying that $\sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$ and

$$\left\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.$$

Theorem 2

(ii) There exists $s_0 \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s \geq s_0$, and $1 < r \leq \infty$, we can find $C > 0$ such that, for every $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, there exist, for each $j \in \mathbb{N}$, $\lambda_j > 0$ and a $(p(\cdot), q(\cdot), r, s)$-atom a_j associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, satisfying that $\sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$ and

$$\left\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.$$

For certain $C > 0$ and for every $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

$$\frac{1}{C}\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \leq \inf \left\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \right\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)},$$

Tools

- $$\|f\|_{p(\cdot),q(\cdot)} = \||f|^{1/r}\|^r_{rp(\cdot),rq(\cdot)} \quad r > 0$$

- $$\|f\|_{p(\cdot),q(\cdot)} = \||f|^{1/r}\|^r_{rp(\cdot),rq(\cdot)} \quad r > 0$$

- Assume that $p, q \in \mathbb{P}_1(0,\infty)$. Then, the maximal function $M_{HL}(f)(x) = \sup_{k\in\mathbb{Z}, y\in x+B_k} \frac{1}{b^k} \int_{y+B_k} |f(z)|dz, \quad x \in \mathbb{R}^n$, is bounded from $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ into itself.

- $$\|f\|_{p(\cdot),q(\cdot)} = \||f|^{1/r}\|^r_{rp(\cdot),rq(\cdot)} \quad r>0$$

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- Assume that $p,q \in \mathbb{P}_1$. For every $r\in(1,\infty)$ there exists $C>0$ such that

$$\left\|\left(\sum_{j\in\mathbb{N}}(M_{HL}(f_j))^r\right)^{1/r}\right\|_{p(\cdot),q(\cdot)} \leq C\left\|\left(\sum_{j\in\mathbb{N}}|f_j|^r\right)^{1/r}\right\|_{p(\cdot),q(\cdot)}, \qquad (1)$$

for each sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $L^1_{loc}(\mathbb{R}^n)$. **(Cruz-Uribe, Martell and Pérez (2011))**

- $$\|f\|_{p(\cdot),q(\cdot)} = \||f|^{1/r}\|^r_{rp(\cdot),rq(\cdot)} \quad r>0$$
- Assume that $p,q \in \mathbb{P}_1(0,\infty)$. Then, the maximal function $M_{HL}(f)(x) = \sup_{k\in\mathbb{Z}, y\in x+B_k} \frac{1}{b^k}\int_{y+B_k}|f(z)|dz, \quad x\in\mathbb{R}^n$, is bounded from $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ into itself.
- Assume that $p,q \in \mathbb{P}_1$. For every $r\in(1,\infty)$ there exists $C>0$ such that

$$\left\| \left(\sum_{j\in\mathbb{N}} (M_{HL}(f_j))^r \right)^{1/r} \right\|_{p(\cdot),q(\cdot)} \leq C \left\| \left(\sum_{j\in\mathbb{N}} |f_j|^r \right)^{1/r} \right\|_{p(\cdot),q(\cdot)}, \qquad (1)$$

  for each sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $L^1_{loc}(\mathbb{R}^n)$. **(Cruz-Uribe, Martell and Pérez (2011))**
- Extrapolation.

- $$\|f\|_{p(\cdot),q(\cdot)} = \||f|^{1/r}\|^r_{rp(\cdot),rq(\cdot)} \quad r > 0$$
- Assume that $p, q \in \mathbb{P}_1(0,\infty)$. Then, the maximal function $M_{HL}(f)(x) = \sup_{k\in\mathbb{Z}, y\in x+B_k} \frac{1}{b^k} \int_{y+B_k} |f(z)|dz, \quad x \in \mathbb{R}^n$, is bounded from $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ into itself.
- Assume that $p, q \in \mathbb{P}_1$. For every $r \in (1,\infty)$ there exists $C > 0$ such that

$$\left\|\left(\sum_{j\in\mathbb{N}}(M_{HL}(f_j))^r\right)^{1/r}\right\|_{p(\cdot),q(\cdot)} \leq C\left\|\left(\sum_{j\in\mathbb{N}}|f_j|^r\right)^{1/r}\right\|_{p(\cdot),q(\cdot)}, \qquad (1)$$

  for each sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $L^1_{loc}(\mathbb{R}^n)$. **(Cruz-Uribe, Martell and Pérez (2011))**
- Extrapolation.
- Rubio de Francia iteration algorithm.

Schema of the proof of Theorem 1

As it was done in **Bownik (2003)** we consider the following maximal functions that will be useful in the sequel. If $K \in \mathbb{Z}$ and $N, L \in \mathbb{N}$ we define, for every $f \in S'(\mathbb{R}^n)$,

$$M_\varphi^{0,K,L,}(f)(x) = \sup_{k \in \mathbb{Z}, k \leq K} |(f * \varphi_k)(x)| \max(1, \rho(A^{-K}x))^{-L}(1 + b^{-k-K})^{-L}, \;\; x \in \mathbb{R}^n,$$

$$M_\varphi^{K,L}(f)(x) = \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in x + B_k} |(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L}(1 + b^{-k-K})^{-L}, \;\; x \in \mathbb{R}^n,$$

$$T_\varphi^{N,K,L}(f)(x) = \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in \mathbb{R}^n} \frac{|(f * \varphi_k)(y)|}{\max(1, \rho(A^{-k}(x-y)))^N} \frac{(1 + b^{-k-K})^{-L}}{\max(1, \rho(A^{-K}y))^L}, \;\; x \in \mathbb{R}^n,$$

$$M_N^{0,K,L}(f) = \sup_{\varphi \in S_N} M_\varphi^{0,K,L}(f),$$

and

$$M_N^{K,L}(f) = \sup_{\varphi \in S_N} M_\varphi^{K,L}(f).$$

Lemma

Let $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$, $r > 0$ *and* $\varphi \in S(\mathbb{R}^n)$. *Then, there exists a constant* $C > 0$ *which does not depend neither on* K, L, N, r *nor* φ *such that, for every* $f \in S'(\mathbb{R}^n)$

$$\left(T_\varphi^{N,K,L}(f)(x)\right)^r \leq C M_{HL}\left(\left(M_\varphi^{K,L}(f)\right)^r\right)(x), \quad x \in \mathbb{R}^n.$$

Lemma

Let $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$, $r > 0$ *and* $\varphi \in S(\mathbb{R}^n)$. *Then, there exists a constant* $C > 0$ *which does not depend neither on* K, L, N, r *nor* φ *such that, for every* $f \in S'(\mathbb{R}^n)$

$$\left(T_\varphi^{N,K,L}(f)(x)\right)^r \leq CM_{HL}\left(\left(M_\varphi^{K,L}(f)\right)^r\right)(x), \quad x \in \mathbb{R}^n.$$

Lemma

Let $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$ *and* $\varphi \in S(\mathbb{R}^n)$. *Assume that* $p, q \in \mathbb{P}_0$. *Then,*

$$\|T_\varphi^{N,K,L}(f)\|_{p(\cdot),q(\cdot)} \leq C\|M_\varphi^{K,L}(f)\|_{p(\cdot),q(\cdot)}, \quad f \in S'(\mathbb{R}^n),$$

where $C > 0$ *does not depend on* (N, K, L, φ).

Lemma. Bownik (2003)

For every $N, L \in \mathbb{N}$, there exists $M_0 \in \mathbb{N}$ satisfying the following property: if $\varphi \in S(\mathbb{R}^n)$ is such that $\int \varphi(x)dx \neq 0$, then there exists $C > 0$ such that, for every $f \in S'(\mathbb{R}^n)$ and $K \in \mathbb{N}$,

$$M_{M_0}^{0,K,L}(f)(x) \leq C T_\varphi^{N,K,L}(f)(x), \quad x \in \mathbb{R}^n.$$

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$$M_{M_0}^{0,K,L}(f)(x) \leq C T_\varphi^{N,K,L}(f)(x), \quad x \in \mathbb{R}^n.$$

Lemma. Bownik (2003)

Let $\varphi \in S(\mathbb{R}^n)$. Then, for every $M, K \in \mathbb{N}$ and $f \in S'(\mathbb{R}^n)$ there exist $L \in \mathbb{N}$ and $C > 0$ such that

$$M_\varphi^{K,L}(f)(x) \leq C \max(1, \rho_A(x))^{-M}, \quad x \in \mathbb{R}^n.$$

Actually, L does not depend on $K \in \mathbb{N}$.

Schema of the proof of Theorem 1

- $\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)} \leq \|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \leq C\|f\|_{H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}$

Schema of the proof of Theorem 1

- $\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)} \leq \|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \leq C\|f\|_{H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}$
- $M_N^0(f)(x) \leq M_N(f)(x) \leq CM_N^0(f)(x), \;\; x \in \mathbb{R}^n$

- $\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)} \leq \|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \leq C\|f\|_{H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}$
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$$\|M_N^0(f)\|_{p(\cdot),q(\cdot)} \sim \|M_N(f)\|_{p(\cdot),q(\cdot)}$$

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- To finish the proof we stablished

$$\|f\|_{H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} = \|M_N^0(f)\|_{p(\cdot),q(\cdot)} \leq C\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \leq c\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)}$$

- $\|M_M^0(f)\|_{p(\cdot),q(\cdot)} \leq C\|M_\varphi(f)\|_{p(\cdot),q(\cdot)}$.
  - $M_M^{0,K,0}(f)(x) \leq CT_\varphi^{0,K,0}(f)(x)$
  - $\|M_M^{0,K,0}(f)\|_{p(\cdot),q(\cdot)} \leq C\|M_\varphi^{K,0}(f)\|_{p(\cdot),q(\cdot)}$

Schema of the proof of Theorem 1

- $\|M_M^0(f)\|_{p(\cdot),q(\cdot)} \leq C\|M_\varphi(f)\|_{p(\cdot),q(\cdot)}$.
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  - $\|M_M^{0,K,0}(f)\|_{p(\cdot),q(\cdot)} \leq C\|M_\varphi^{K,0}(f)\|_{p(\cdot),q(\cdot)}$
- $\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \leq C\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)}$.

Calderón-Zygmud decomposition

Let $f \in S'(\mathbb{R}^n)$ satisfying that $|\{x \in \mathbb{R}^n : \; M_N f(x) > \lambda\}| < \infty$. Assume that $p, q \in \mathbb{P}_0$. There exists $s_0 \in \mathbb{N}$, such that, for every $s \in \mathbb{N}$, $s \geq s_0$, and each $N \in \mathbb{N}$, $N > max\{N_0, s\}$, where N_0 is defined in Theorem 1, the following two properties holds.

(i) Let $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $\lambda > 0$. If $f = g + \sum_{i\in\mathbb{N}} b_i$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N f$ of height λ and degree s, then the series $\sum_{i\in\mathbb{N}} b_i$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

(ii) Suppose that $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and that, for every $j \in \mathbb{Z}$, $f = g_j + \sum_{i\in\mathbb{N}} b_{i,j}$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N f$ of height 2^j and degree s. Then, $(g_j)_{j\in\mathbb{Z}} \subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $(g_j)_{j\in\mathbb{Z}}$ converges to f, as $j \to +\infty$, in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

The good part is in $L^1_{loc}(\mathbb{R}^n)$

If $f \in S'(\mathbb{R}^n)$, $\lambda > 0$, $s, N \in \mathbb{N}$, $N \geq 2$ and $s < N$, and $f = g + \sum_{i \in \mathbb{N}} b_i$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N(f)$ of height λ and degree s, then $g \in L^1_{loc}(\mathbb{R}^n)$.

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Density of a space of functions

Assume that $p, q \in \mathbb{P}_0$. Then, $L^1_{loc}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is a dense subspace of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

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Convergence of the good parts

Assume that $p, q \in \mathbb{P}_0$, and $f \in L^1_{loc}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. For every $j \in \mathbb{N}$, $f = g_j + \sum_{i\in\mathbb{N}} b_{i,j}$ is the anisotropic Calderón-Zygmund decomposition of f associated to $M_N(f)$ of height 2^j and degree s, with $s, N \in \mathbb{N}$, $s \geq s_0$ and $N > max\{s, N_0\}$, where N_0 is as in Theorem 1 and s_0 is as in Proposition. Then, $g_j \to 0$, as $j \to -\infty$, in $S'(\mathbb{R}^n)$.

$H^{p(\cdot),q(\cdot)}_{atom}(\mathbb{R}^n, A) \subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

- $f_{\ell,m} = \sum_{j=\ell}^{m} \lambda_j a_j$,
- $\|M_\varphi(f_{\ell,m})\|_{p(\cdot),q(\cdot)} \leq C \left(\left\| \sum_{j=\ell}^{m} \lambda_j M_\varphi(a_j) \chi_{x_j+B_{\ell_j+\omega}} \right\|_{p(\cdot),q(\cdot)} + \left\| \sum_{j=\ell}^{m} \lambda_j M_\varphi(a_j) \chi_{x_j+B^c_{\ell_j+\omega}} \right\|_{p(\cdot),q(\cdot)} \right) = I_1 + I_2$

I_1 estimate

- $M_\varphi(a_j)(x) \leq \|a_j\|_\infty \|\varphi\|_1 \leq C \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)}, \; x \in \mathbb{R}^n$.
- $g_j = \chi_{x_j+B_{\ell_j}} (\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \lambda_j)^\alpha$
- $M_{HL} g_j(x) \geq b^{-\omega} (\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \lambda_j)^\alpha, \quad x \in x_j + B_{\ell_j+\omega}$

Making use of the vectorial inequality for M_{HL} we get

$$I_1 \leq C \left\| \sum_{j=\ell}^{m} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}$$

I_2 estimate

- $M_\varphi(a)(x) \leq C \frac{1}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} (M_{HL}(\chi_{z+B_k})(x))^\gamma, \ x \notin z + B_{k+\omega}$

Proceeding as in the case of I_1 we get

$$I_2 \leq C \left\| \sum_{j=\ell}^{m} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}$$

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Proceeding as in the case of I_1 we get

$$I_2 \leq C \left\| \sum_{j=\ell}^{m} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}$$

Having been established that $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete, the inclusion is proved

$H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A) \subset H^{p(\cdot),q(\cdot)}_{atom}(\mathbb{R}^n, A)$

- $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A) \cap L^1_{loc}(\mathbb{R}^n)$
- Let $j \in \mathbb{Z}$. We define $\Omega_j = \{x \in \mathbb{R}^n : \; M_N(f)(x) > 2^j\}$
- $f = \sum_{j\in\mathbb{Z}}(g_{j+1} - g_j)$, in $S'(\mathbb{R}^n)$.
- $g_{j+1} - g_j = \sum_{i\in\mathbb{N}} h_{i,j}$, in $S'(\mathbb{R}^n)$,
- $h_{i,j} = (f - P^j_i)\zeta^j_i - \sum_{k\in\mathbb{N}}((f - P^{j+1}_k)\zeta^j_i - P^{j+1}_{i,k})\zeta^{j+1}_k$, $\; i \in \mathbb{N}$.
- $a_{i,j} = h_{i,j} 2^{-j} C_0^{-1} \|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|^{-1}_{p(\cdot),q(\cdot)}$ is a $(p(\cdot), q(\cdot), \infty, s)$-atom

Finally

$$f = \sum_{i\in\mathbb{N}, j\in\mathbb{Z}} \lambda_{i,j} a_{i,j} \;\text{ in }\; S'(\mathbb{R}^n), \qquad (2)$$

where $\lambda_{i,j} = 2^j C_0 \|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|_{p(\cdot),q(\cdot)}$, for every $i \in \mathbb{N}$, $j \in \mathbb{Z}$.

$H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A) \subset H^{p(\cdot),q(\cdot)}_{atom}(\mathbb{R}^n, A)$

Let $1 < r < \infty$ and let $p, q \in \mathbb{P}_0$. There exists $s_0 \in \mathbb{N}$ satisfying that if $s \in \mathbb{N}$, $s \geq s_0$, we can find $C > 0$ for which, for every $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, there exist, for each $j \in \mathbb{N}$, $\lambda_j > 0$ and a $(p(\cdot), q(\cdot), r, s)$-atom a_j associated with some $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, such that

$$\left\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)} \leq C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)},$$

and $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$.

Consequence of the fact that if a is a $(p(\cdot), q(\cdot), \infty, s)$-atom, then a is a $(p(\cdot), q(\cdot), r, s)$-atom

$H^{p(\cdot),q(\cdot)}_{atom}(\mathbb{R}^n, A) \subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

- If a is a $(p(\cdot), q(\cdot), r, s_0)$-atom, then $a \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

$H^{p(\cdot),q(\cdot)}_{atom}(\mathbb{R}^n, A) \subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

- If a is a $(p(\cdot), q(\cdot), r, s_0)$-atom, then $a \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$

Strömberg and Torchinsky (1989) (anisotropic version)

- $\left\| \sum_{k\in\mathbb{N}} \lambda_k \chi_{x_k+B_{\ell_k+\ell}} \right\|_{L^p(\mathbb{R}^n,\nu)} \leq Cb^{\ell\delta} \left\| \sum_{k\in\mathbb{N}} \lambda_k \chi_{x_k+B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n,\nu)}.$
- $\|\sum_{k\in\mathbb{N}} \lambda_k a_k\|_{L^p(\mathbb{R}^n,\nu)} \leq C \left\| \sum_{k\in\mathbb{N}} \lambda_k \chi_{x_k+B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n,\nu)}.$
- $\|f\|_{H^p(\mathbb{R}^n,\nu,A)} \leq C \left\| \sum_{k=1}^{\infty} \lambda_k \chi_{x_k+B_{\ell_k}} \right\|_{L^p(\mathbb{R}^n,\nu)}.$
- $\|f\|_{H^{p_0}(\mathbb{R}^n,\nu,A)} \leq C \left\| \sum_{k\in\mathbb{N}} \lambda_k \|\chi_{x_k+B_{\ell_k}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_k+B_{\ell_k}} \right\|_{L^{p_0}(\mathbb{R}^n,\nu)}.$

Proposition

Let $p, q \in \mathbb{P}_0$ *being* $p(0) \leq q(0)$. *There exist* $s_0 \in \mathbb{N}$ *and* $r_0 > 1$ *such that, for every* $r \geq r_0$ *we can find* $C > 0$ *satisfying that if, for every* $j \in \mathbb{N}$, $\lambda_j > 0$ *and* a_j *is a* $(p(\cdot), q(\cdot), r, s_0)$*-atom associated with* $x_j \in \mathbb{R}^n$ *and* $\ell_j \in \mathbb{Z}$ *such that*

$$\sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n),$$

then $f = \sum_{j\in\mathbb{N}} \lambda_j a_j \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, *and*

$$\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \leq C \left\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}.$$

Dual spaces of $L_{p(),q(\cdot)}$

- The description of these spaces.
- Are the Hardy-Littlewood Maximal function bounded on them?

- The study of operatos on these spaces.

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THANK YOU VERY MUCH

- $\theta \in C^\infty(\mathbb{R}^n)$ verifies that supp $\theta \subset B_\omega$, $0 \leq \theta \leq 1$, and $\theta = 1$ on B_0. We define, for every $j \in \mathbb{N}$,

$$\theta_j(x) = \theta(A^{-\ell_j}(x - x_j)), \quad x \in \mathbb{R}^n,$$

and, for every $i \in \mathbb{N}$,

$$\zeta_i(x) = \begin{cases} \theta_i(x)/(\sum_{j\in\mathbb{N}} \theta_j(x)), & x \in \Omega_\lambda, \\ 0, & x \in \Omega_\lambda^c. \end{cases}$$

The sequence $\{\zeta_i\}_{i\in\mathbb{N}}$ is a smooth partition of unity associated with the covering $\{x_i + B_{\ell_i+\omega}\}_{i\in\mathbb{N}}$ of Ω.

- Let $i, s \in \mathbb{N}$. By $\mathcal{P}_s$ we denote the linear space of polynomials in $\mathbb{R}^n$ with degree at most s. $\mathcal{P}_s$ is endowed with the norm $\| \; \|_{i,s}$ defined by

$$\|P\|_{i,s} = \left(\frac{1}{\int \zeta_i} \int_{\mathbb{R}^n} |P(x)|^2 \zeta_i(x) dx \right)^{1/2}, \quad P \in \mathcal{P}_s.$$

Thus $(\mathcal{P}_s, \| \; \|_{i,s})$ is a Hilbert space. We consider on $\mathcal{P}_s$ the functional $T_{f,i,s}$ given by

$$T_{f,i,s}(Q) = \frac{1}{\int \zeta_i} < f, Q\zeta_i >, \quad Q \in \mathcal{P}_s.$$

$T_{f,i,s}$ is continuous in $(\mathcal{P}_s, \| \; \|_{i,s})$ and there exists $P_{f,i,s} \in \mathcal{P}_s$ such that

$$T_{f,i,s}(Q) = \frac{1}{\int \zeta_i} \int_{\mathbb{R}^n} P_{f,i,s}(x) Q(x) \zeta_i(x) dx, \quad Q \in \mathcal{P}_s.$$

To simplify we write P_i to refer to $P_{f,i,s}$.

- we define $M^0_{HL}(h) = |h|$ and, for every $i \in \mathbb{N}$, $i \geq 1$, $M^i_{HL}(h) = M_{HL} \circ M^{i-1}_{HL}(h)$. We consider

$$R(h) = \sum_{i=0}^{\infty} \frac{M^i_{HL}(h)}{2^i \|M_{HL}\|^i_{(\alpha p(\cdot))',(\alpha q(\cdot))'}}.$$

  We have that
  (i) $|h| \leq R(h)$;
  (ii) R is bounded from $\mathcal{L}^{(\alpha p(\cdot))',(\alpha q(\cdot))'}(\mathbb{R}^n)$ into itself and $\|R(h)\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'} \leq 2\|h\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'}$;
  (iii) $R(h) \in \mathcal{A}_1(\mathbb{R}^n, A)$ and $[R(h)]_{\mathcal{A}_1(\mathbb{R}^n,A)} \leq 2\|M_{HL}\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'}$. Hence, there exists $\beta_0 > 1$ such that $R(h) \in RH_{\beta_0}(\mathbb{R}^n, A)$.

- $f_k = \sum_{j=0}^{k} \lambda_j a_j$, then $f_k \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

- $\|M_\varphi(f_k)\|^{1/\alpha}_{p(\cdot),q(\cdot)} = \|(M_\varphi(f_k))^{1/\alpha}\|_{\alpha p(\cdot),\alpha q(\cdot)} \leq C \sup_h \int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} h(x)dx,$

$$
\begin{aligned}
\int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} h(x)dx &\leq \int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} R(h)(x)dx \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}}(x) \right)^{1/\alpha} R(h)(x)dx \\
&\leq C \left\| \left(\sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right)^{1/\alpha} \right\|_{\alpha p(\cdot),\alpha q(\cdot)} \|R(h)\|_{(\alpha p(\cdot)
\\
&\leq C \left\| \sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}^{1/\alpha} \|h\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'} \\
&\leq C \left\| \sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}^{1/\alpha}.
\end{aligned}
$$

Hence, we obtain

$$
\|f_k\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \leq C \left\| \sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}^{-1} \chi_{x_j+B_{\ell_j}} \right\|_{p(\cdot),q(\cdot)}.
$$