Hardy-Littlewood maximal operator in weighted Lorentz spaces

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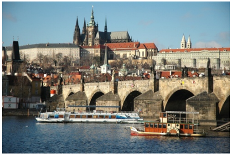


Classical results

Hardy-Littlewood maximal operator

Definition

The Hardy-Littlewood maximal operator is defined by:

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)| dy,$$

where I is an interval of the real line and the supremum is considered over all intervals containing $x \in \mathbb{R}$.

Problem

Characterize the weights $u \in L^1_{loc}(\mathbb{R})$ so that $M : L^p(u) \to L^p(u)$, that is; find necessary and sufficient conditions on the weight u so that

$$||Mf||_{L^p(u)} \lesssim ||f||_{L^p(u)}.$$

(I) The case of weighted Lebesgue spaces $L^p(u)$

Recall $\quad u \in A_p : \quad \sup_{I \subset \mathbb{R}} \left(\frac{1}{|I|} \int_I u(x)dx \right) \left(\frac{1}{|I|} \int_I u^{-1/(p-1)}(x)dx \right)^{p-1} < \infty.$

Theorem (Muckenhoupt 1972)

For $p > 1$, the following statements are equivalent:

(i) $M : L^p(u) \to L^p(u)$;

(ii) $u \in A_p$;

(iii) $\dfrac{u(I)}{u(S)} \lesssim \left(\dfrac{|I|}{|S|} \right)^{p-\varepsilon}$, *for every interval I and $S \subset I$ and some $\varepsilon > 0$.*

Weak-type Lebesgue space $L^{p,\infty}(u)$

Recall that the space $L^{p,\infty}(u)$ is defined by

$$\left\{ f \in \mathcal{M}(\mathbb{R}) : ||f||^p_{L^{p,\infty}(u)} = \sup_{\lambda>0} \lambda^p u(\{x \in \mathbb{R} : |f(x)| > \lambda\}) < \infty \right\}.$$

It has been also studied the boundedness $M : L^p(u) \to L^{p,\infty}(u)$, equivalently

$$||Mf||_{L^{p,\infty}(u)} \lesssim ||f||_{L^p(u)}.$$

Theorem (Muckenhoupt 1972)

For $p > 1$, the following statements are equivalent:

(i) $M : L^p(u) \to L^p(u)$;

(ii) $M : L^p(u) \to L^{p,\infty}(u)$;

(iii) $u \in A_p$.

Motivation for introducing the classical Lorentz spaces

Note that:

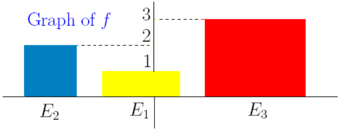
$$\|f\|_{L^p}^p = \int_{\mathbb{R}} |f(x)|^p dx = \int_0^\infty p\lambda^{p-1}|\{x \in \mathbb{R} : |f(x)| > \lambda\}|d\lambda = \int_0^\infty (f^*(t))^p dt,$$

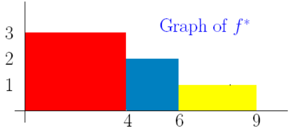
where

$$f^*(t) = \inf\{s > 0 : |\{|f| > s\}| \leq t\},$$

and

$$\|f\|_{L^{p,\infty}}^p = \sup_{\lambda > 0} \lambda^p |\{x \in \mathbb{R} : |f(x)| > \lambda\}| = \sup_{t>0} t(f^*)^p(t).$$





Classical Lorentz spaces

Definition (Lorentz, 1950)

Let $p > 0$, $w \in L^1_{loc}(\mathbb{R}^+)$. *The classical Lorentz spaces are:*

$$\Lambda^p(w) = \{f \in \mathcal{M}(\mathbb{R}) : ||f||^p_{\Lambda^p(w)} = \int_0^\infty (f^*(t))^p w(t)dt < \infty\},$$

$$\Lambda^{p,\infty}(w) = \{f \in \mathcal{M}(\mathbb{R}) : ||f||^p_{\Lambda^{p,\infty}(w)} = \sup_{\lambda>0} \lambda^p W(|\{x : f(x) > \lambda\}|) < \infty\},$$

where $f^*(t) = \inf\{s > 0 : |\{x \in \mathbb{R} : |f(x)| > s\}| \leq t\}$ *and* $W(t) = \int_0^t w$.

Examples:

(*i*) If $w = 1$, we recover $\Lambda^p(1) = L^p$ and $\Lambda^{p,\infty}(1) = L^{p,\infty}$;

(*ii*) If $w = t^{q/p-1}$, we obtain $L^{p,q}$.

It is known that $(Mf)^*(t) \approx Pf^*(t)$, where

$$Pf(t) = \frac{1}{t}\int_0^t f(r)dr = \int_1^\infty f(t/s)\frac{ds}{s^2} = \int_1^\infty E_s f(t)\frac{ds}{s^2},$$

where $E_s f(t) = f(t/s)$. For $s \in [1,\infty)$, define

$$\|E_s\|^p_{\Lambda^p(w)\to\Lambda^p(w)} = \sup_{\|f^*\|_{L^p(w)}\leq 1} \|E_s f^*\|^p_{L^p(w)} = \sup_{t>0}\frac{W(st)}{W(t)} =: \overline{W}(s).$$

By Minkowski inequality we obtain

$$\begin{aligned}||M||_{\Lambda^p(w)\to\Lambda^p(w)} &= \sup_{\|f\|_{\Lambda^p(w)}\leq 1}\|Mf\|_{\Lambda^p(w)} \approx \sup_{\|f^*\|_{L^p(w)}\leq 1}\|Pf^*\|_{L^p(w)} \\ &= \sup_{\|f^*\|_{L^p(w)}\leq 1}\left\|\int_1^\infty E_s f^*\frac{ds}{s^2}\right\|_{L^p(w)} \leq \int_1^\infty \overline{W}^{1/p}(s)\frac{ds}{s^2}.\end{aligned}$$

Proposition

If $\displaystyle\int_1^\infty \overline{W}^{1/p}(s)\frac{ds}{s^2} < \infty$, *then* $M : \Lambda^p(w) \to \Lambda^p(w)$.

Theorem (Lorentz-Shimogaki 1960, Ariño-Muckenhoupt 1990)

Let $p > 1$. Then the following are equivalent:

(i) $M : \Lambda^p(w) \to \Lambda^p(w)$;

(ii) $M : \Lambda^p(w) \to \Lambda^{p,\infty}(w)$;

(iii) $\displaystyle\int_1^\infty \overline{W}^{1/p}(s)\frac{ds}{s^2} < \infty$;

(iv) *There exists* $s \in [1,\infty)$ *so that* $\overline{W}(s) < s^p$;

(v) $\displaystyle\frac{W(t)}{W(r)} \lesssim \left(\frac{t}{r}\right)^{p-\varepsilon}$, $t \geq r$, *for some* $\varepsilon > 0$.

Define

$$w \in B_p \qquad \text{if} \qquad \frac{W(t)}{W(r)} \lesssim \left(\frac{t}{r}\right)^{p-\varepsilon}, \quad t \geq r,$$

for some $\varepsilon > 0$.

Weighted Lorentz spaces

Weighted Lorentz spaces

Definition (Lorentz, 1950)

Let $p > 0$, $u \in L^1_{loc}(\mathbb{R})$, $w \in L^1_{loc}(\mathbb{R}^+)$. *The weighted Lorentz spaces are:*

$$\Lambda^p_u(w) = \{f \in \mathcal{M}(\mathbb{R}) : ||f||^p_{\Lambda^p_u(w)} = \int_0^\infty (f^*_u(t))^p w(t)dt < \infty\},$$

$$\Lambda^{p,\infty}_u(w) = \{f \in \mathcal{M}(\mathbb{R}) : ||f||_{\Lambda^{p,\infty}_u(w)} = \sup_{\lambda>0} \lambda^p W(u(\{x : f(x) > \lambda\})) < \infty\},$$

where

$$f^*_u(t) = \inf\{s > 0 : u(\{x \in \mathbb{R} : |f(x)| > s\}) \leq t\},$$

$$u(E) = \int_E u(x)dx \quad \text{and} \quad W(t) = \int_0^t w(s)ds.$$

Examples:

(*i*) If $w = 1$, we recover $\Lambda^p_u(1) = L^p(u)$ and $\Lambda^{p,\infty}_u(1) = L^{p,\infty}(u)$;

(*ii*) If $u = 1$, we obtain $\Lambda^p(w)$ and $\Lambda^{p,\infty}(w)$.

Characterization of $M : \Lambda_u^p(w) \rightarrow \Lambda_u^{p,\infty}(w)$

Problem I

Find necessary and sufficient conditions on $u \in L^1_{loc}(\mathbb{R})$ and $w \in L^1_{loc}(\mathbb{R}^+)$:

$$M : \Lambda_u^p(w) \rightarrow \Lambda_u^p(w),$$

equivalently $||Mf||_{\Lambda_u^p(w)} \leq C||f||_{\Lambda_u^p(w)}$.

Problem II

Find necessary and sufficient conditions on $u \in L^1_{loc}(\mathbb{R})$ and $w \in L^1_{loc}(\mathbb{R}^+)$:

$$M : \Lambda_u^p(w) \rightarrow \Lambda_u^{p,\infty}(w),$$

equivalently $||Mf||_{\Lambda_u^{p,\infty}(w)} \leq C||f||_{\Lambda_u^p(w)}$.

Theorem (Carro-Raposo-Soria 2007)

Let $p > 1$. The following statements are equivalent:

- $M : \Lambda_u^p(w) \to \Lambda_u^p(w)$;
- *There exists $\varepsilon > 0$ so that:*

$$\frac{W(u(\bigcup_j I_j))}{W(u(\bigcup_j S_j))} \lesssim \max_{1 \le j \le J} \left(\frac{|I_j|}{|S_j|} \right)^{p-\varepsilon}, \tag{1}$$

for every finite family of disjoint intervals $(I_j)_{j=1}^J$, and every family of measurable sets $(S_j)_{j=1}^J$, so that $S_j \subset I_j$, for every j.

If $w = 1$, the above condition is equivalent to A_p : $\dfrac{u(I)}{u(S)} \lesssim \left(\dfrac{|I|}{|S|} \right)^{p-\varepsilon}$.

If $u = 1$, it is equivalent to B_p : $\dfrac{W(t)}{W(r)} \lesssim \left(\dfrac{t}{r} \right)^{p-\varepsilon}$, $t \ge r$.

Open

Characterization of the weak-type boundedness $M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$.

Weak-type boundedness of M

Theorem (A-Antezana-Carro 2016)

Let $p > 1$, then

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Leftrightarrow M : \Lambda_u^p(w) \to \Lambda_u^p(w).$$

Known case $w = 1$: The implication

$$M : L^p(u) \to L^{p,\infty}(u) \Rightarrow M : L^p(u) \to L^p(u)$$

is proved using reverse Hölder's inequality: if $u \in A_p$, then $\exists\, \gamma > 0$ so that

$$\left(\frac{1}{|I|} \int_I u^{1+\gamma}(t)dt \right)^{\frac{1}{1+\gamma}} \leq C \frac{1}{|I|} \int_I u(t)dt.$$

Then, if $u \in A_p$ we have that $u \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. Using interpolation

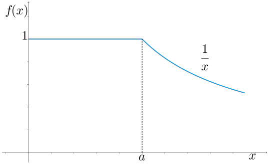
$$M : L^{p-\varepsilon}(u) \to L^{p-\varepsilon,\infty}(u) \quad \text{and} \quad M : L^\infty \to L^\infty$$

we get the strong type boundedness.

Weak-type boundedness of M

Case $u = 1$: $M : \Lambda^p(w) \to \Lambda^{p,\infty}(w) \Rightarrow M : \Lambda^p(w) \to \Lambda^p(w)$. Let $s > 1$ and

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le a; \\ \frac{a}{x}, & \text{if } a \le x \le sa; \\ 0, & \text{if } x > sa. \end{cases}$$



Since $(Mf)^* \approx Pf$, by the hypothesis

$$W(|\{x : Pf(x) > y\}|) \le C\frac{1}{y^p}\int_0^\infty f^p(x)w(x)dx. \tag{2}$$

Note that $Pf(as) = \dfrac{1+\log s}{s}$. Taking $y = \dfrac{(1+\log s)}{s}$ we have

$$W(|(0,as)|) \lesssim W(|\{x : Pf(x) > (1+\log s)/s\}|) \le C(1+\log s)^{1-p}s^p W(|(0,a)|).$$

Hence,

$$\frac{W(as)}{W(a)} \le C(1+\log s)^{1-p}s^p. \tag{3}$$

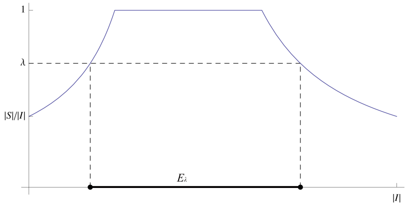
Thus $M : \Lambda^p(w) \to \Lambda^p(w)$.

Weak-type boundedness of M

Theorem (A-Antezana-Carro 2015)

$$M:\Lambda^p_u(w)\to\Lambda^{p,\infty}_u(w)\Rightarrow M:\Lambda^p_u(w)\to\Lambda^p_u(w).$$

General case: Let $S\subset I\subset\mathbb{R}$. Construct $f_{S,I}$, so that for every $\lambda\in[|S|/|I|,1]$



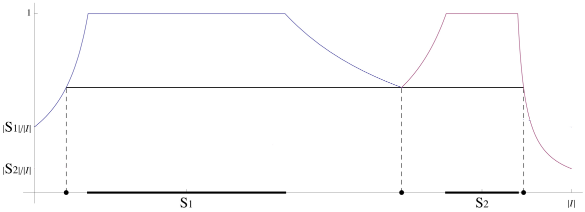
so that $|S|=\lambda|E_\lambda|$, where $E_\lambda=\{x:f_{S,I}(x)>\lambda\}$. Then, for $s=\frac{|I|}{|S|}$

$$||Mf_{S,I}||^p_{\Lambda^{p,\infty}_u(w)}\le C\,||f_{S,I}||^p_{\Lambda^p_u(w)}\Rightarrow\frac{W(u(I))}{W(u(S))}\le C\,(1+\log s)^{1-p}s^p,$$

which "implies" that $M:\Lambda^p_u(w)\to\Lambda^p_u(w)$.

Weak-type boundedness of M

If $S = S_1 \cup S_2 \subset I \subset \mathbb{R}$, the function $f_{S,I}$ can be:



Several dimensions: We consider the function

$$\chi_{\{M\chi_S > \lambda\}} M\chi_S,$$

for any measurable set $S \subset \mathbb{R}^d$.

Possible problem to study

- Recall that the Riesz transforms are given by

$$R_j f(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy,$$

  for $j = 1, \ldots, d$, whenever they are well defined. Study the boundedness of the Riesz transforms on weighted Lorentz spaces.

Some references

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Thank you for your attention!